Neutral Geometry Theorems

Pasch's Theorem/Axiom. If a line ℓ intersects one of the sides of a triangle, then it must intersect one of the other two sides.

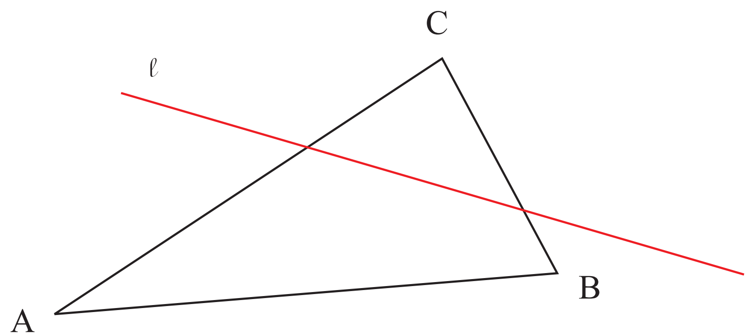

Figure 1: Pasch's theorem

Equivalently: If the line ℓ intersects side $\overline{AC}$ of $\triangle ABC$ and contains neither A nor B then it intersects $\overline{AB}$ or $\overline{CB}$.

Crossbar Theorem. If $\angle BAC$ is an angle, and ℓ is a ray emanating from A and $\overline{DE}$ is a segment intersecting rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then ℓ intersects $\overline{DE}$.

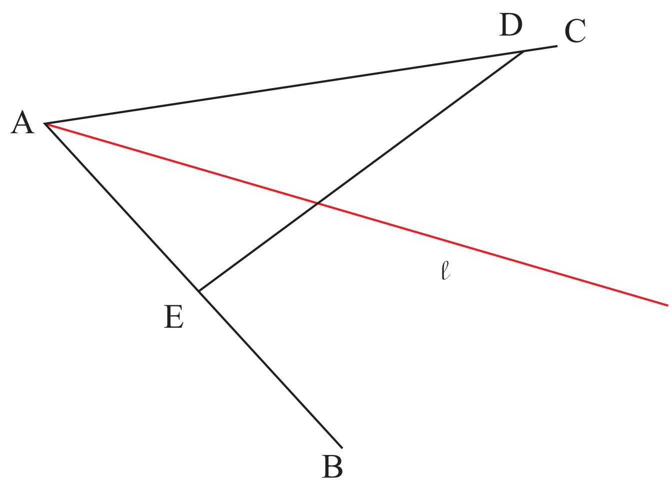

Figure 2: Crossbar theorem

Theorem ASA.

Proof. Let $\triangle ABC$ have two angles and the included side respectively congruent to $\triangle DEF$ with:

$$\begin{aligned} \angle CAB &\cong \angle FDE \\ \overline{AB} &\cong \overline{DE} \\ \angle ABC &\cong \angle DEF. \end{aligned}$$

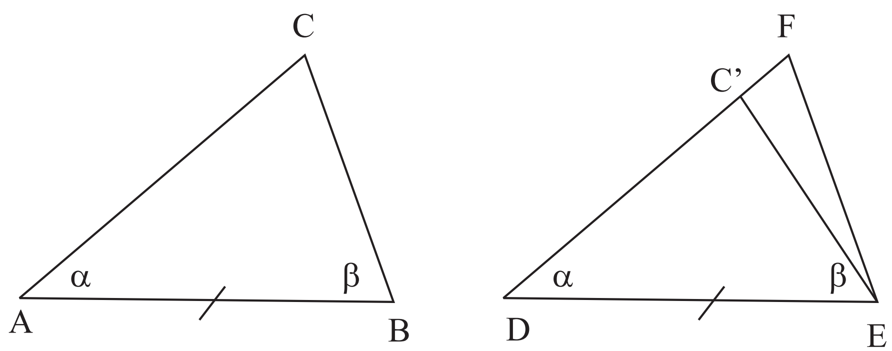

Figure 3: ASA

Suppose that the triangles are not congruent. Then (by SAS axiom) $m(\overline{AC}) \neq m(\overline{DF})$. So we may assume, without loss of generality, that $m(\overline{AC}) < m(\overline{DF})$. Then there is a point C' between D and F so that $\overline{AC} \cong \overline{DC'}$. Then by SAS axiom, $\triangle ABC \cong \triangle DEC'$. So $\angle ABC \cong \angle DEC'$; but $\overline{C'F}$ lies in the interior of $\angle DEF$ and so $m(\angle DEC') < m(\angle DEF)$. This contradicts $\angle DEC' \cong \angle ABC \cong \angle DEF$. $\square$

Theorem SSS. Suppose that two triangles have three corresponding sides congruent to each other. Then the triangles themselves are congruent to each other.

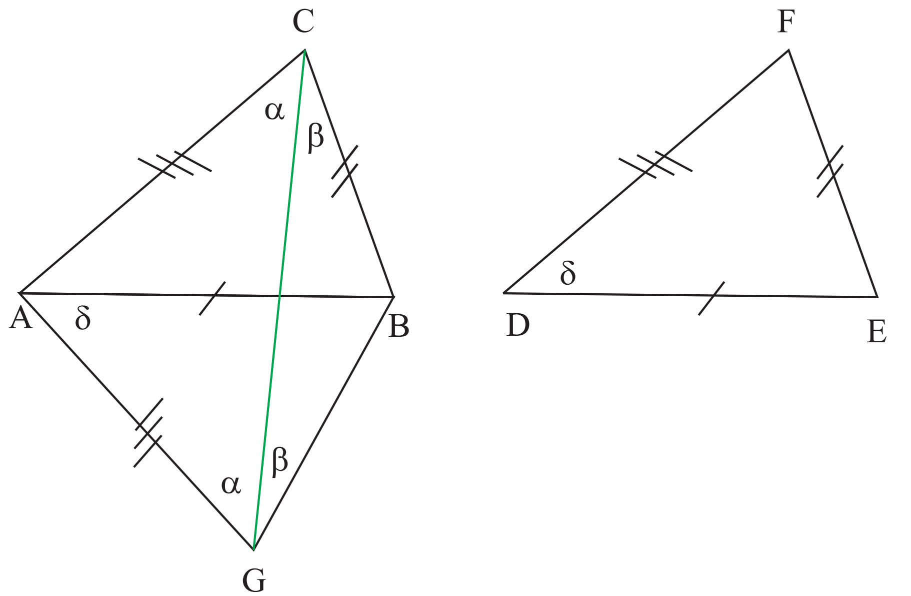

Figure 4: SSS

Proof. Let $\triangle ABC$ have three sides respectively congruent to three sides of $\triangle DEF$ with:

$$\begin{aligned} \overline{AB} &\cong \overline{DE} \\ \overline{BC} &\cong \overline{EF} \\ \overline{CA} &\cong \overline{FD}. \end{aligned}$$

Let G be a point on the opposite side of $\overleftrightarrow{AB}$ than C so that $\angle BAG \cong \angle EDF$ and so that $\overline{AG} \cong \overline{DF}$. Then, by SAS, $\triangle BAG \cong \triangle EDF$. We consider first the case where $\overline{CG}$ is interior to $\angle ACB$ and therefore intersects $\overline{AB}$. Since $\triangle CAG$ is isosceles we have $\angle ACG \cong \angle AGC$. Similarly $\triangle CBG$ is isosceles and we also have $\angle BCG \cong \angle BGC$. Therefore, using the summation property, $\angle ACB \cong \angle AGB$ and we have $\triangle ACB \cong \triangle AGB$ by SAS. And since, $\triangle BAG \cong \triangle EDF$, we have $\triangle BAC \cong \triangle EDF$. $\square$

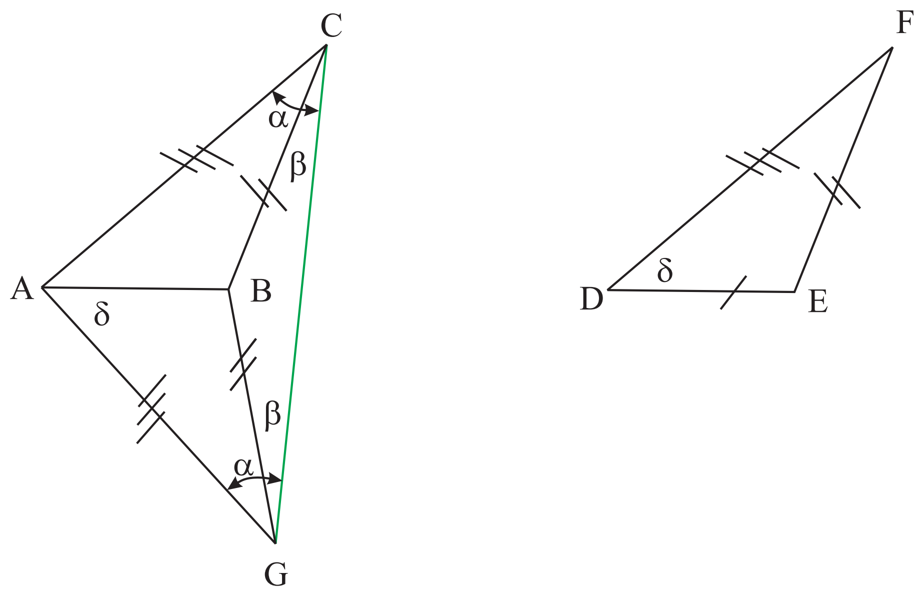

Figure 5: SSS Case 2

Case 2 is very similar except that two angle measurements subtract rather than add.

Theorem [Alternate Interior Angles.] Suppose that ℓ and m are two line cut by a transversal line n. Let A and B be the intersection points of n with ℓ and m respectively. Let C be a point on ℓ and not on n, let D be a point on m the other side of n than C. Suppose that the alternate interior angles $\angle CAB$ and $\angle DBA$ are congruent. Then the lines ℓ and m are parallel.

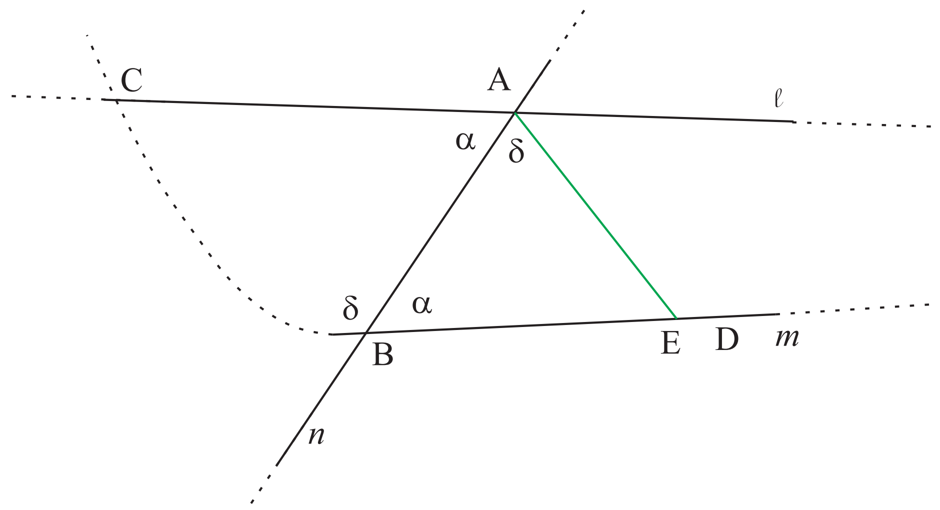

Figure 6: Alternate Interior Angles

Proof. Suppose not and that ℓ and m intersect. Without loss of generality, we may assume that ℓ and m intersect at the point C. Then let E be a point of m on the same side of m as D so that $\overline{BE} \cong \overline{AC}$.

Then by SAS $\triangle CAB \cong \triangle EBA$.

So, $\angle ABC \cong \angle BAE$; but $\angle ABE$ is supplementary to $\angle ABC$. So $\angle BAE$ is supplementary to $\angle BAC$ since these angle are congruent. And so the point E must also lie on line ℓ since $\angle BAE$ together with $\angle BAC$ form a straight angle.

Then lines ℓ and m have the two points C and E in common and so $\ell = m$ which contradicts the hypothesis that these are two different line.

Therefore ℓ does not intersect m and so the lines are parallel. □

Exterior Angle Theorem. Suppose that $\triangle ABC$ is a triangle and $\overline{AB}$ is extended to the ray $\overrightarrow{AD}$ (with B between A and D). Then the measure of $\angle DBC$ is greater then either of the two interior angles $\angle BAC$ and $\angle BCA$. [Note $\angle DBC$ is said to be an angle exterior to $\angle ABC$.]

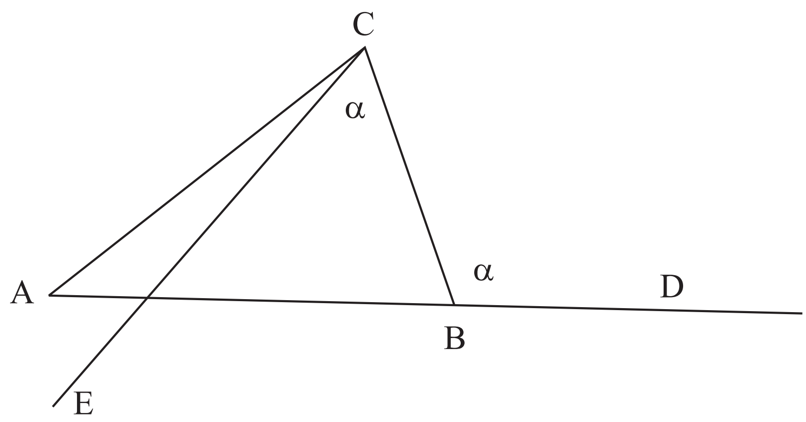

Figure 7: Exterior Angle Theorem

Proof. Suppose the theorem is not true and that, without loss of generality, assume $m(\angle DBC) \leq \angle m(ACB)$. Now if $m(\angle DBC) = \angle m(ACB)$ then by the alternate interior angle theorem, $\overleftrightarrow{AC}$ would be parallel to $\overleftrightarrow{AB}$, which is not possible. Therefore, $m(\angle DBC) < m(\angle ACB)$. Let $\overrightarrow{CE}$ be a ray emanating from C so that $\angle BCE \cong \angle DBC$; furthermore, since $m(\angle DBC) \leq \angle m(BCA)$, E can be chosen to be in the interior of $\angle BCA$. By the alternate interior angle theorem $\overleftrightarrow{CE}$ is parallel to $\overleftrightarrow{AB}$ and by the crossbar theorem $\overrightarrow{CE}$ intersects $\overline{AB}$. Which is a contradiction. □