Picture and argument related to Newton's lemmas.

In the beginning of the Principia, Newton states some lemmas about ultimate ratios. Here are the first two from Section I. Section I is titled, "The method of first and last rations of quantities, by the help of which we demonstrates the propositions that follow" (the propositions are about the motion of bodies).

Lemma 1. *Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.*

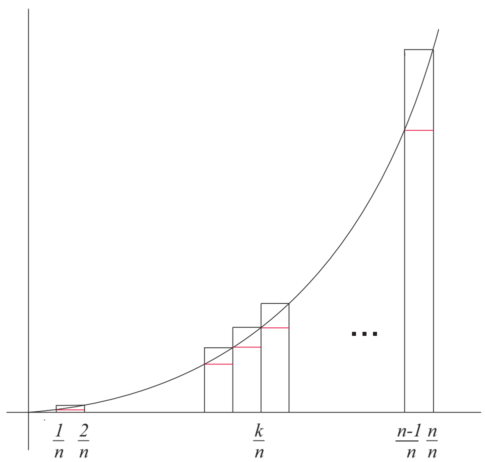

Figure 1: Upper and Lower Rectangles for $y = x^2$.

Let's do an example illustrating Newton's idea:

For the function $y = x^2$ we're interested in the area under the curve, above the x-axis and between $x = 0$ and $x = 1$. The interval $[0, 1]$ is partitioned into n subintervals each of length $\frac{1}{n}$. We will need the following formula.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let A denote the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$. We know that the area is between the upper rectangles (in black) and the lower rectangles (in red). The area of the k^{th} upper rectangle is height $\times$ base and is $\left(\frac{k}{n}\right)^2 \times \frac{1}{n}$ and the k^{th} lower rectangle is $\left(\frac{k-1}{n}\right)^2 \times \frac{1}{n}$. Since the area is between these two collections of rectangles we have [I'm skipping some steps],

$$\begin{aligned} \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} &< A < \sum_{k=1}^{n} \frac{k^2}{n^3} \\ \frac{(n-1)(n)(2n-1)}{6n^3} &< A < \frac{n(n+1)(2n+1)}{6n^3}. \end{aligned}$$

For all positive integers n. We can express the two fractions as follows:

$$\begin{aligned} \frac{(n-1)(n)(2n-1)}{6n^3} &= \frac{2n^3 - 3n^2 + n}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \\ \frac{n(n+1)(2n+1)}{6n^3} &= \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}. \end{aligned}$$

It is not too hard to see that the first expression above approaches "nearer" to $\frac{1}{3}$ "than by any given difference" and similarly the second expression also approaches nearer to $\frac{1}{3}$. So using Newton's lemma we have:

$$\frac{1}{3} \leq A \leq \frac{1}{3}.$$

Exercise 1: For any given difference ϵ find an integer N_ϵ so that for $n > N_\epsilon$

$$\frac{1}{3} - \left(\frac{1}{2n} + \frac{1}{6n^2}\right)$$

is nearer to $\frac{1}{3}$ than the given difference.

Exercise 2: For any given difference ϵ find an integer N_ϵ so that for $n > N_\epsilon$

$$\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

is nearer to $\frac{1}{3}$ than the given difference.