**Pascal's Triangle**.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & \vdots & & & & & & \vdots & & & &
\end{array}
$$

I draw the triangle in its modern incarnation, its row shifted from Pascal's presentation. Pascal wrote what he had discovered about this configuration of numbers in his monograph *Triangle Arithmétique*. The top row is designated the $0^{\text{th}}$ row, the zeroth row. The next one is the first row and so on. The elements are numbered from left to right (or from right to left - it doesn't make any difference because of the symmetry of the triangle - this is Consequence V in his book); and their position is labeled 0, 1, 2, . . . and so on. So that the zeroth entry in each row is always a 1. For example, the second entry in the fifth row is 10; the first entry in the $n^{\text{th}}$ is $n$. The modern notation for the $r^{\text{th}}$ entry in the $n^{\text{th}}$ row is $\binom{n}{r}$. The $r^{\text{th}}$ entry of the $(n+1)^{\text{th}}$ row of the triangle is calculated from the $n^{\text{th}}$ row by adding the entries in the previous row that are north-west and north-east of it.

In closed form:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

In order for the formula to make sense when $r = 0$ or $r = n$ we use the notational convention that $0! = 1$ by definition. Following are identities for you to work on regarding the elements of Pascal's Triangle.

1. First we prove the correctness of the closed form representation for each integer $r = 0, 1, 2, \ldots, n, n+1$:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Note: for $r = 0$ we assume $\binom{n}{-1} = 0$ and for $r = n+1$ we assume $\binom{n}{n+1} = 0$. This is equivalent to assuming that there are 0's before the first and after the last element of each line.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 0 & & & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & 0 & & 1 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 1 & & 0 & & & & \\
 & & & 0 & & 1 & & 2 & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 3 & & 3 & & 1 & & 0 & & \\
 & 0 & & 1 & & 4 & & 6 & & 4 & & 1 & & 0 & \\
0 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & 0 \\
 & & & & \vdots & & & & & & \vdots & & & &
\end{array}
$$

2. Proving the following identity shows why the term $\binom{n}{r}$ is called a binomial coefficient:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r.$$

3. In Pascal's monograph, Consequence II is the following:

$$\binom{n}{r} = \sum_{i=1}^{n-r+1} \binom{n-i}{r-1}.$$

4.) Pascal's Consequence VIII calculates the sum of all the elements in a row:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

5.) Pascal's Consequence XII is of particular historical significance; it is the first time that induction in the modern formulation was stated and used to prove a statement; using the modern formulation it says that for all positive integers $n$ and $r = 1, 0, \ldots, n-1$:

$$\frac{\binom{n}{r}}{\binom{n}{r+1}} = \frac{r+1}{n-r}.$$

6.) Finally, for an application to probability: show that the probability of getting exactly $r$ sixes in $n$ throws of a fair die is:

$$\text{Prob } = \binom{n}{r} \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{n-r}.$$