A Study of the Pentagon

We assume that the sum of the angles of a triangle is 180°. From this fact, and using the symmetry of the figure, we can obtain the measurements of the indicated angles in the picture below of the regular pentagon.

$$\begin{aligned}
\mathrm{m}(\alpha) &= 108^\circ \\
\mathrm{m}(\alpha) - \mathrm{m}(\beta) &= 36^\circ \\
\mathrm{m}(\beta) &= 72^\circ \\
\mathrm{m}(\gamma) &= 72^\circ \\
\mathrm{m}(\delta) &= 36^\circ \\
\mathrm{m}(\epsilon) &= 36^\circ.
\end{aligned}$$

Next, we use the properties of isosceles triangles to obtain the following congruencies.

$$\begin{aligned}
\overline{AB} &\cong \overline{AD} \\
\overline{BD} &\cong \overline{DC} \\
\overline{AE} &\cong \overline{DC} \\
\overline{FD} &\cong \overline{BD}
\end{aligned}$$

We assume that there is some quantity x that measure both AB and AC an integer number of times. Say the integers m and n respectively. Then:

$$\begin{aligned}
\mathrm{m}(\overline{AB}) &= m \\
\mathrm{m}(\overline{AC}) &= n \\
\mathrm{m}(\overline{DC}) &= n - m \\
\mathrm{m}(\overline{ED}) &= 2m - n \\
\mathrm{m}(\overline{FD}) &= n - m.
\end{aligned}$$

This means that the quantity x also measures, an integer number of times, both $\overline{ED}$ and $\overline{FD}$ with measurements $2m - n$ and $n - m$ respectively.

This should lead to an "A-ha" moment: this leads to a contradiction. So the assumption that the quantities $\overline{AB}$ and $\overline{AC}$ are commensurable is false.

The "A-ha" realization is that the quantity x measures the diagonal and side of the smaller pentagon respectively, an integer number times. By the same reasoning this process continues to smaller and smaller pentagons (each lying inside the previous one) until eventually the pentagon is smaller than the length x.

Problem: Show geometrically that the diagonal and side of a square are incommensurable. [Hint: a helpful picture follows that of the pentagon. In the picture, $\overline{AC}$ is constructed congruent to $\overline{AF}$.]

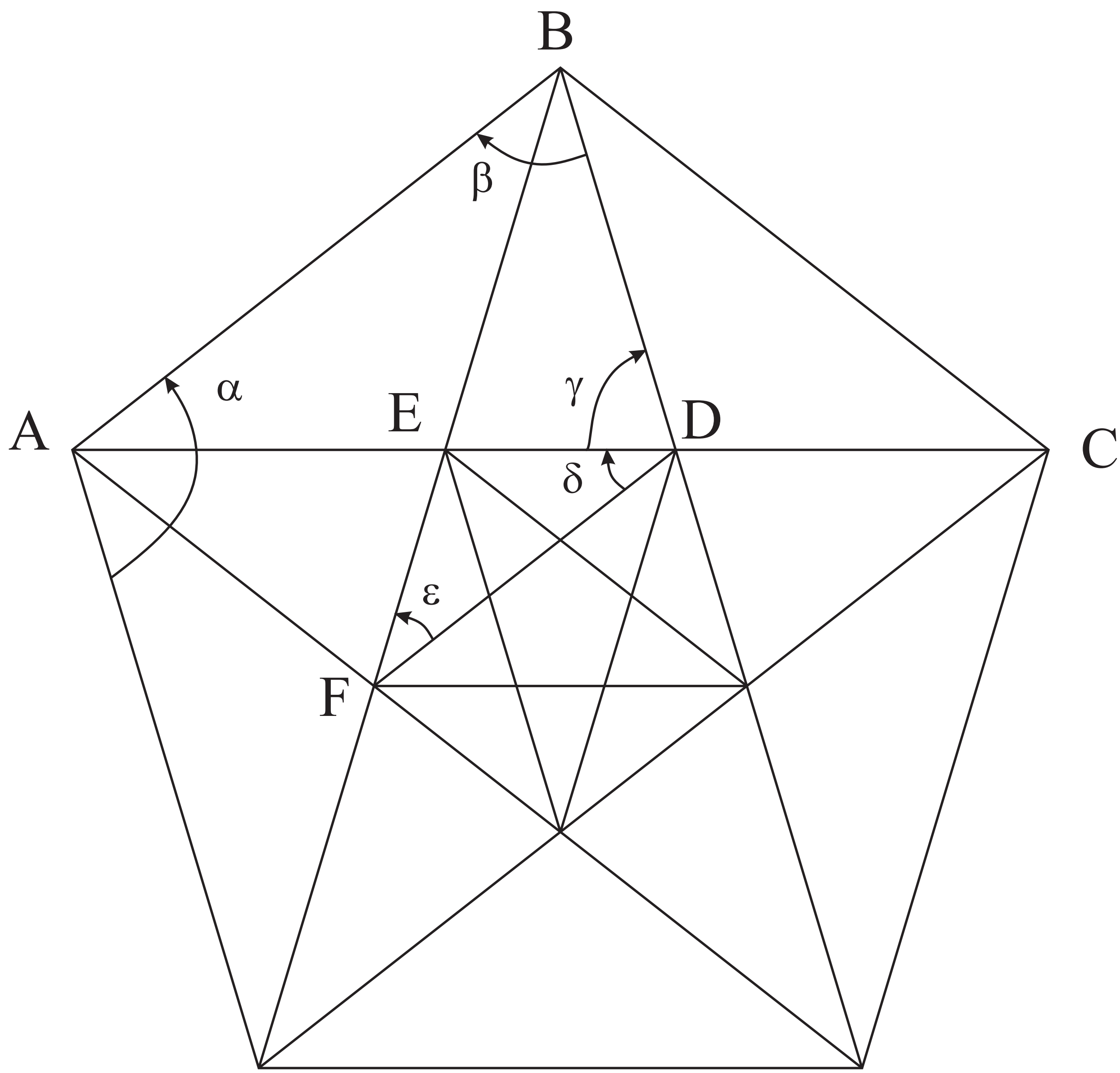

Figure 1: Regular Pentagon

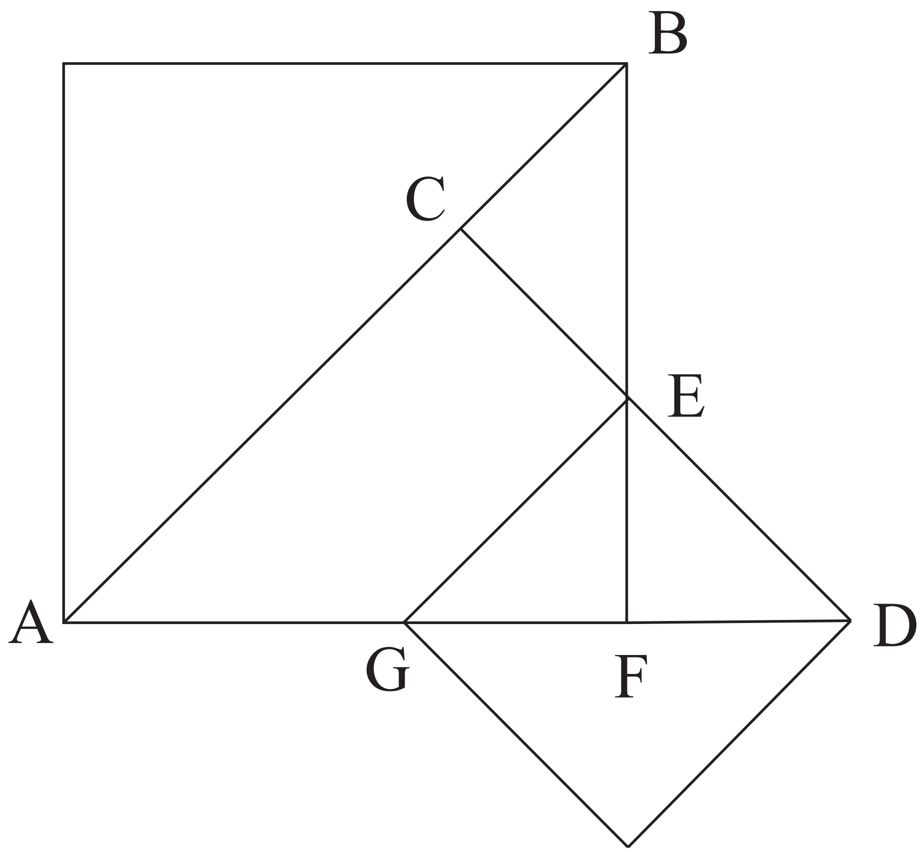

Figure 2: Square