Solving the Cubic and Quartic Equations

Cardano - Tartaglia method for solving the cubic equation.

Part I. We solve the equation $x^3 + mx = n$.

We'll start by looking at $(a-b)^3$ and putting that in the form of the above equation:

$$\begin{aligned}(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\\ (a-b)^3 + 3a^2b - 3ab^2 &= a^3 - b^3\\ (a-b)^3 + 3ab(a-b) &= a^3 - b^3\\ x^3 + mx &= n\end{aligned}$$

Then letting $x = a - b$ and equating like terms we get a system of two equations:

$$3ab = m \quad \text{and} \quad a^3 - b^3 = n.$$

Solving for a and b yields:

$$\begin{aligned}3ab &= m\\ a &= \frac{m}{3b}\\ a^3 - b^3 &= n\\ \frac{m^3}{2b^3} - b^3 &= n\\ m^3 - 2b^6 &= 2nb^3.\end{aligned}$$

This last equation is quadratic in b^3 and so can be solved for b; then once b is known a can be calculated. Once a and b are known then $x = a - b$ is calculated.

Part II. We solve the reduce the general equation to one that can be done using the method of part I. Consider $x^3 + ax^2 + bx = c$. We want to "remove" the x^2 terms so we try a change of variable - we substitute $t + r$ for x and find an r that eliminates the quadratic term.

$$\begin{aligned}x^3 + ax^2 + bx &= c\\ (t+r)^3 + a(t+r)^2 + b(t+r) &= c\\ t^3 + 3rt^2 + 3r^2t + r^3 + at^2 + 2atr + ar^2 + bt + br &= c\\ t^3 + (3r+a)t^2 + (3r^2 + 2ar)t + +bt + r^3 + ar^2 + br &= c.\end{aligned}$$

So if we pick $r = -\frac{a}{3}$ then the x^2 term vanishes and we can solve the new equation in t using the technique of part I. Once that is done our solution is obtained by letting $x = t + r$ and solving the new cubic for t and then adding r to t to get x.

Ferrari's method for solving the quartic equation.

Part III. The general quartic. Consider the equation $x^4 + ax^3 + bx^2 + cx = d$. First we get rid of the troublesome x^3 term using the same trick that was used in the cubic. Let $x = t + r$ then:

$$\begin{aligned}
(t+r)^4 + a(t+r)^3 + b(t+r)^2 + c(t+r) &= d \\
t^4 + 4rt^3 + 6r^2t^2 + 4r^3t + r^4 + at^3 + 3art^2 + 3ar^2t + ar^3 + & \\
b(t+r)^2 + c(t+r) &= d \\
t^4 + (4r+a)t^3 + (\text{ quadratic in } t) &= d.
\end{aligned}$$

So letting $r = -\frac{a}{4}$ eliminates the cubic term and we now have an equation of the form $x^4 + px^2 + qx + r = 0$. First we complete a square in x^2.

$$\begin{aligned}
x^4 + px^2 + qx + r &= 0 \\
x^4 + 2px^2 + qx + r &= px^2 \\
x^4 + 2px^2 + p^2 + qx + r &= px^2 + p^2 \\
(x^2 + p)^2 + qx + r &= px^2 + p^2 \\
(x^2 + p)^2 &= px^2 + p^2 - qx - r.
\end{aligned}$$

This equation is in the form of a square on the left equal to something, $z^2 =$ stuff. So we'll complete the square. Let y be such that when we add $2zy + y^2$ to both sides, the right becomes a perfect square, let $z = x^2 + p$ so we would have:

$$\begin{aligned}
z^2 &= px^2 + p^2 - qx - r \\
z^2 + 2yz + y^2 &= 2yz + y^2 + px^2 + p^2 - qx - r \\
(z+y)^2 &= 2yz + y^2 + px^2 + p^2 - qx - r \\
(z+y)^2 &= 2y(x^2 + p) + y^2 + px^2 + p^2 - qx - r \\
(z+y)^2 &= (2y + p)x^2 - qx - (2yp + y^2 + p^2 - r).
\end{aligned}$$

Now the right side is a perfect square whenever the discriminant $B^2 - 4AC$ of the quadratic $Ax^2 + Bx + C$ is zero. So that gives us the following equation for y

$$q^2 - 4(2y + p)(2yp + y^2 + p^2 - r) \;\; = \;\; 0$$

which gives us a cubic equation in y which can be solved using the techniques above (and which must always have at least one real root.)

So now the equation is in the following form which is easily solved:

$$(x^2 + p + y)^2 \;\; = \;\; (ax + b)^2.$$