Incommensurability of the diagonal and side of the square.

Problem: Show geometrically that the diagonal and side of a square are incommensurable.

Assume that we have a square pictured below with diagonal $\overline{AB}$ and sides $\overline{AF}$ and $\overline{FB}$. We assume that there is some quantity x that measure both AF and AB an integer number of times. Say the integers m and n respectively. We mark off $\overline{AC}$ congruent to $\overline{AF}$ and $\overline{CD}$ is constructed perpendicular to $\overline{AB}$.

Then using properties of isosceles triangles and elementary geometry, we have:

$$\begin{aligned} \mathrm{m}(\overline{AF}) &= m \\ \mathrm{m}(\overline{AB}) &= n \\ \mathrm{m}(\overline{AC}) &= m \\ \mathrm{m}(\overline{CE}) &= n-m \\ \mathrm{m}(\overline{ED}) &= 2m-n \\ \mathrm{m}(\overline{FD}) &= n-m \\ \mathrm{m}(\overline{GD}) &= 2(n-m). \end{aligned}$$

This means that the quantity x also measures, an integer number of times, both $\overline{ED}$ and $\overline{GD}$ with measurements $2m-n$ and $2(n-m)$ respectively.

Again, this leads to a contradiction. So the assumption that the quantities $\overline{AB}$ and $\overline{AF}$ are commensurable is false.

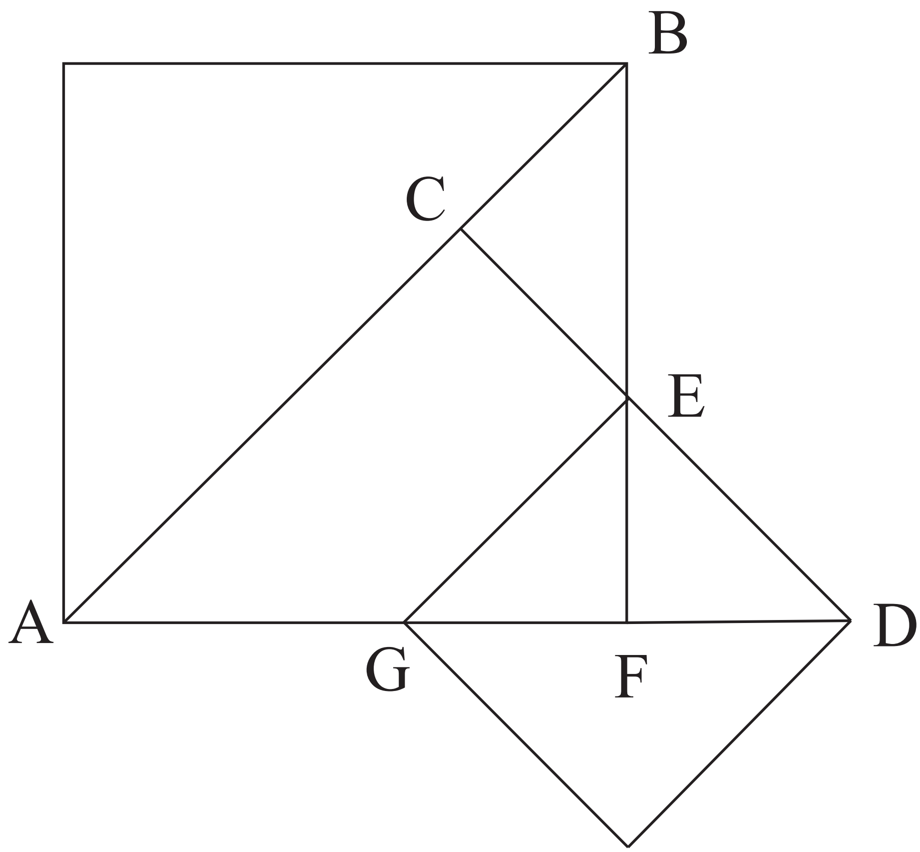

Figure 1: Square