The Definition of the Limit.

Following is the definition of limit due to Weierstrass following the work of Bolzano (1817) and Cauchy (1821).

For ease of understanding assume that $f : \mathbb{R} \to \mathbb{R}$.

Definition 1. *The limit of a function f.*

$$L = \lim_{x \to p} f(x)$$

means that for each positive number ϵ there is a number δ_ϵ so that:

$$\textit{If } 0 < |x - p| < \delta_\epsilon \quad \textit{then} \quad |f(x) - L| < \epsilon.$$

Equivalently: the limit of a function $f(x)$ as x approaches p is L means that for each $\epsilon > 0$ there is a $\delta_\epsilon > 0$ (which is typically dependent on ϵ) so that for all values of x within δ_ϵ of p the corresponding $f(x)$ value is within ϵ of L.

One can compare this to an archery contest between Robin Hood and a sequence of contestants trying to hit a bull's eye on a target. Each contestant makes the target bull's eye smaller than the previous one (the ϵ's) but Robin Hood can still hit inside the bull's eye by adjusting the angle of arrow within a certain range (the δ_ϵ used to adjust his aim). The angle adjustments have to be finer and finer as the contestants narrow the size of the bull's eye. And he does this without every having to hit the exact center. The "exact center" is the point that Robin Hood is aiming for, and that point is the mathematical limit of the $\epsilon - \delta$ definition.

Based on this idea I've created a "limit game" between archery challengers (player $\mathcal{E}$) and Robin Hood (player $\mathcal{D}$):

The limit game.

The Limit game is played on the reals $\mathbb{R}$ and starts with a function f, a point p and a number L, the function need not be defined at p. For example:

$$f(x) = \frac{\sqrt{x} - 2}{x - 4}$$

where p is the number 4 and the function is not defined at 4.

There are two players: Player $\mathcal{E}$ and player $\mathcal{D}$.

Player $\mathcal{E}$ goes first and picks a number $\epsilon > 0$. Then player $\mathcal{D}$ must pick a number $\delta > 0$; the number δ must have the property that if t is within δ of p (but not equal to p), then $f(t)$ is within ϵ of L. Algebraically:

$$\text{If } 0 < |p - t| < \delta \text{ then } |L - f(t)| < \epsilon.$$

Then the play continues turn by turn, in the n^{th} turn of play: player $\mathcal{E}$ picks a number ϵ_n then player $\mathcal{D}$ must pick a number $\delta_n > 0$ that satisfies the rules of the game ... and so on.

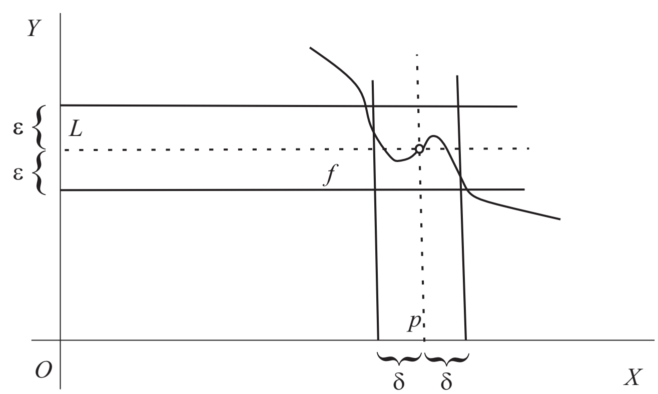

Figure 1: Player $\mathcal{D}$ selecting a good move.

Player $\mathcal{E}$ wins at the n^{th} step of the game if player $\mathcal{D}$ cannot find a number that satisfies the rules of the game. Player $\mathcal{D}$ wins if $\mathcal{D}$ can always find a $\delta_n > 0$ that satisfies the rules of the game.

Using the Robin Hood metaphor, $\mathcal{E}$ sets up a target for $\mathcal{D}$ to shoot at and with each turn $\mathcal{E}$ is allowed to make the target smaller and smaller (but never zero).

Since this game has the possibility of being an infinite game, what is required for player $\mathcal{D}$ to win is for $\mathcal{D}$ to provide a "winning strategy." (Alternatively, one can assume that each player makes their n^{th} move in the time interval between $\frac{1}{n}$ and $\frac{1}{n+1}$ minutes before midnight and $\mathcal{D}$ wins if the game didn't end before midnight with a win on $\mathcal{E}$'s part.)

Examples. For the following "Gameboards", determine who will win and give a winning strategy for that player:

a.)	$f(x) = 7x$	$p = 3$	$L = 21$
b.)	$f(x) = 7x$	$p = 3$	$L = 22$
c.)	$f(x) = x^2$	$p = 2$	$L = 4$
d.)	$f(x) = x^2$	$p = \frac{1}{2}$	$L = \frac{1}{2}$
e.)	$f(x) = x^2$	$p = \frac{1}{2}$	$L = \frac{1}{4}$
f.)	$f(x) = \frac{x^2-4}{x-2}$	$p = 2$	$L = 4$
g.)	$f(x) = \frac{x^2-4}{x-2}$	$p = 2$	$L = 3.5$
h.)	$f(x) = \frac{x^3-8}{x-2}$	$p = 2$	$L =?$
i.)	$f(x) = \frac{\sqrt{x}-2}{x-4}$	$p = 4$	$L = \frac{1}{2}$
j.)	$f(x) = \frac{\sqrt{x}-2}{x-4}$	$p = 4$	$L = \frac{1}{4}$
k.)	$f(x) = \frac{x^2+5x-14}{x-2}$	$p = 2$	$L =?$
l.)	$f(x) = \frac{1}{x}$	$p = 0$	$L = 2$
m.)	$f(x) = \frac{1}{x}$	$p = 0$	$L = 1000$
n.)	$f(x) = \frac{1}{x}$	$p = 0$	$L > 0$
o.)	$f(x) = \frac{45-5x^2}{3-x}$	$p = 3$	$L = 30.$

Theorem: The function $f : (a, b) \to \mathbb{R}$ is continuous at $(p, f(p))$ if and only if $p \in (a, b)$ and player $\mathcal{D}$ has a winning strategy for the point p and $L = f(p)$.

Lets do an example. We'll do exercise (o.) above.
We claim that a wining strategy for $\mathcal{D}$ is as follows: If $\mathcal{E}$ selects ϵ then player $\mathcal{D}$ selects δ_ϵ to be any positive number so that $\delta_\epsilon \leq \frac{\epsilon}{5}$. We now prove that this strategy works:

Proof. Suppose that $\epsilon > 0$ and $\delta_\epsilon \leq \frac{\epsilon}{5}$. Then whenever x is such that $|x-3| < \delta_\epsilon$ we have:

$$\begin{aligned}
\left|\frac{45-5x^2}{3-x} - 30\right| &= \left|\frac{5(3+x)(3-x)}{3-x} - 30\right| \\
&= |15+5x-30| \\
&= |15+5x-30| \\
&= |5x-15| \\
&= |5(x-3)| \\
&< 5\delta_\epsilon = 5\frac{\epsilon}{5} = \epsilon.
\end{aligned}$$

Here's the scratch work that I used to figure out the δ that would work I want

$$\left|\frac{45-5x^2}{3-x} - 30\right| < \epsilon.$$

So I did the same algebra to obtain:

$$\begin{aligned}
\left|\frac{45-5x^2}{3-x} - 30\right| &= \left|\frac{5(3+x)(3-x)}{3-x} - 30\right| \\
&= \vdots \\
&= |5(x-3)|.
\end{aligned}$$

So I need $|5(x-3)| < \epsilon$; solving for $|x-3|$:

$$\begin{aligned}
|5(x-3)| &< \epsilon \\
|(x-3)| &< \frac{\epsilon}{5}.
\end{aligned}$$

So selecting $\delta_\epsilon \leq \frac{\epsilon}{5}$ should work; and indeed it does as the proof shows. Notice that δ is dependent on ϵ. □

Let's see how Weierstrass's definition can be used to rigorously calculate a derivative. First we'll use Leibnitz' technique to derive the derivative of $f(x) = 5x^2$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y(x+dx) - y(x)}{dx} \\
&= \frac{y(x+dx) - y(x)}{dx} \\
&= \frac{5(x+dx)^2 - 5 \cdot x^2}{dx} \\
&= \frac{5(x^2 + 2xdx + (dx)^2) - 5x^2}{dx} \\
&= \frac{10xdx + 5(dx)^2}{dx} \\
&= 10x + 5dx \\
&= 10x.
\end{aligned}$$

Where the last step follows from the fact that dx and hence $5dx$ is an infinitesimal which when added to something doesn't change its value.

Now I'll use the definition of limit due to Weierstrass to prove that this is correct. The limit definition of the derivative is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

Where I use $h = \Delta x$ to make the limit look more like what we saw in Calculus I.

We want to prove that:

$$\lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = 10x.$$

Proof. First observe that the number x is some fixed number, like $x = 3$. So as the limit game's player $\mathcal{E}$ tells us, suppose $\epsilon > 0$. Then player $\mathcal{D}$ needs to pick a δ_ϵ. Player $\mathcal{D}$'s strategy is to pick $\delta_\epsilon = \frac{\epsilon}{5}$. [How they decided this will be clear when we examine the calculations.] Then when $0 < |h - 0| < \delta$ we have:

$$\begin{aligned}
\left|\frac{y(x+h)-y(x)}{h}-L\right| &= \left|\frac{5(x+h)^2-5\cdot x^2}{h}-10x\right| \\
&= \left|\frac{5(x^2+2xh+h^2)-5x^2}{h}-10x\right| \\
&= \left|\frac{5x^2+10xh+5h^2-5x^2}{h}-10x\right| \\
&= \left|\frac{10xh+5h^2}{h}-10x\right| \\
&= |10x+5h-10x| \\
&= |5h| = 5|h| < 5\delta = 5\frac{\epsilon}{5} = \epsilon.
\end{aligned}$$

Therefore for our choice of δ, if $0 < |h-0| < \delta$ then

$$\left|\frac{5(x+h)^2-5\cdot x^2}{h}-10x\right| < \epsilon.$$

□