

## Descartes' Method for finding the Tangent to a Curve.

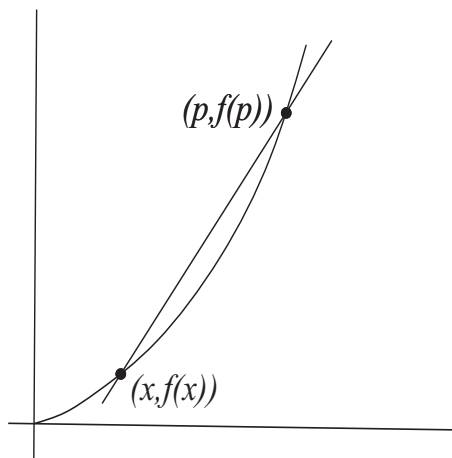


Figure 1:  $y = f(x)$

René Descartes used the following method to find the slope of a line tangent to a curve. For a given curve  $y = f(x)$  he considered the line of some slope  $m$  containing the point  $(p, f(p))$  to which he wants to find the tangent. The equation of the line containing  $(p, f(p))$  of slope  $m$  is:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - f(p) &= m(x - p) \\ y &= f(p) + mx - mp. \end{aligned}$$

This line intersects the curve at  $(p, f(p))$  and possibly at other points. This means that  $p$  satisfies the equation  $f(x) = f(p) + mx - mp$ . And so  $p$  is a root of the polynomial (or more generally the function)  $P(x) = f(x) - f(p) - mx + mp$ . His insight is that if  $m$  could be selected so that  $p$  is a root of multiplicity at least 2, then that value of  $m$  would be the slope of the tangent line. I will show how this works with the special case of the cubic:  $y = x^3$ . This is the curve  $f(x) = x^3$ . We will find the slope of the line tangent to the curve  $f(x) = x^3$  at the point  $(p, p^3)$ .

We consider the polynomial  $P(x) = x^3 - p^3 - mx + mp$ . We know that  $p$  is a root of the polynomial so  $(x - p)$  is a factor of the polynomial. Use long

division to factor the polynomial and you should obtain:

$$\begin{aligned}x^3 - p^3 - mx + mp &= (x - p)(x^2 + xp + p^2) - m(x - p) \\ &= (x - p)(x^2 + xp + p^2 - m).\end{aligned}$$

In order for  $p$  to be a double root of the polynomial  $P(x)$  it must be a root of the second term of the product on the right above. So that says that, substituting  $p$  for  $x$ ,

$$x^2 + xp + p^2 - m = p^2 + pp + p^2 - m = 0.$$

Solving for  $m$  gives us:

$$\begin{aligned}p^2 + pp + p^2 - m &= 0 \\ 3p^2 &= m.\end{aligned}$$

So the desired slope is  $m = 3p^2$ . Notice that this matches what we learned in calculus. Once you have the slope, the equation of the line tangent to the curve at the point  $(p, p^3)$  can be easily calculated.