Euler Constant

Euler started with the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

to obtain

$$\begin{aligned}
\ln\left(1+\frac{1}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \dots \\
\ln\left(\frac{1+x}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \dots \\
\frac{1}{x} &= \ln\left(\frac{1+x}{x}\right) + \frac{1}{2x^2} - \frac{1}{3x^3} + \frac{1}{4x^4} - \dots
\end{aligned}$$

Substitute $x = 1, 2, 3, \dots, n$:

$$\begin{aligned}
1 &= \ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \\
\frac{1}{2} &= \ln\left(\frac{3}{2}\right) + \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \dots \\
\frac{1}{3} &= \ln\left(\frac{4}{3}\right) + \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \dots \\
\frac{1}{4} &= \ln\left(\frac{5}{4}\right) + \frac{1}{2 \cdot 4^2} - \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} - \dots \\
\vdots &= \vdots \\
\frac{1}{n} &= \ln\left(\frac{n+1}{n}\right) + \frac{1}{2 \cdot n^2} - \frac{1}{3 \cdot n^3} + \frac{1}{4 \cdot n^4} - \dots
\end{aligned}$$

using the property of logarithms and adding

$$\begin{aligned}
1 &= \ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \\
\frac{1}{2} &= \ln 3 - \ln 2 + \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \dots \\
\frac{1}{3} &= \ln 4 - \ln 3 + \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \dots \\
\frac{1}{4} &= \ln 5 - \ln 4 + \frac{1}{2 \cdot 4^2} - \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} - \dots \\
\vdots &= \vdots \\
\frac{1}{n} &= \ln(n+1) - \ln n + \frac{1}{2 \cdot n^2} - \frac{1}{3 \cdot n^3} + \frac{1}{4 \cdot n^4} - \dots
\end{aligned}$$

yields

$$
\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} &= \ln(n+1)+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{i^2}+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}+\frac{1}{4}\sum_{i=1}^{n}\frac{1}{i^4}+\dots \\
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} &= \ln(n+1)+C.
\end{aligned}
$$

The constant C is called Euler's constant and in modern notation it is denoted by γ. It will be the following limit:

$$
\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)\right) \\
&= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)+\ln n-\ln n\right) \\
&= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln n-\ln\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln n-0\right) \\
&= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln n\right) \\
&\approx 0.5772156649015328606065120900824024310421 \text{ from Wikipedia}
\end{aligned}
$$

Euler first mentioned the constant in a 1734 paper. Is it not known if it is transcendental, nor even irrational. It is related to the derivative Ψ of the gamma function:

$$-\gamma = \Gamma'(1) = \Psi(1).$$