Fundamental Theorem of Calculus

The earliest version was by James Gregory in 1668. Isaac Barrow knew a more generalized version of the theorem. Isaac Newton completed the development of a theory regarding the inverse relationship between the tangent problem and the area problem. Later Gottfried Leibniz rediscovered the theorem that was in Newton's unpublished work using significantly different notation; he (and to a certain extent Newton) used a concept of infinitesimal quantities to obtain his results.

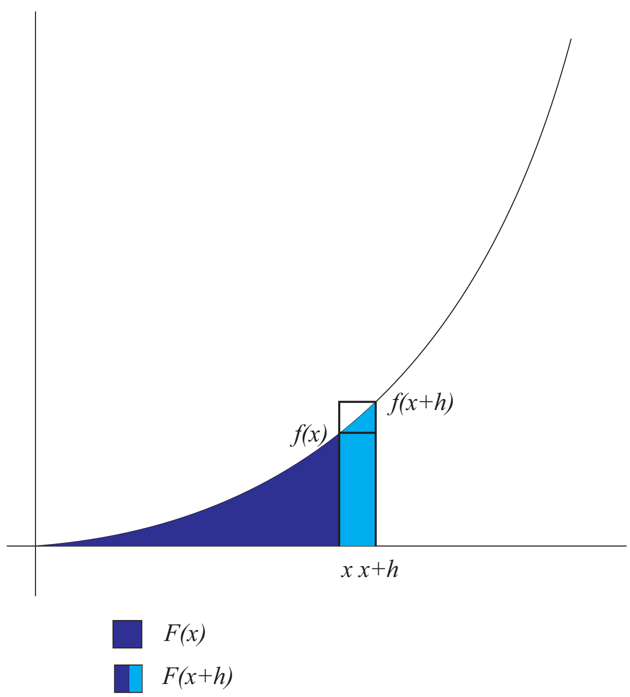

Figure 1. Fundamental Theorem of Calculus

Suppose that $y = f(x)$ is a function; for the sake of our picture I'll assume that it's an increasing function. We also assume that the function is continuous.

We want to calculate the area under the graph bounded by x-axis and the vertical lines through $x = a$ and $x = b$. We let $F(x)$ denote the area under the graph bounded by x-axis from $x = 0$ to the vertical line through x. This area is indicated in dark blue. The area between the vertical lines at x and $x+h$ is indicated in light blue; note that I've drawn the picture for the case of $h>0$. So the area in light blue is $F(x+h)-F(x)$. The area of the small rectangle (base $\times$ height) is $h \cdot f(x)$ and is less than the area in light blue which is less than the area of the big rectangle $h \cdot f(x+h)$. So we have

$$h \cdot f(x) < F(x+h) - F(x) < h \cdot f(x+h).$$

Dividing by h gives us:

$$f(x) < \frac{F(x+h)-F(x)}{h} < f(x+h).$$

Using Newton's terminology, the ultimate ratio of the middle potion of the inequality is $F'(x)$; and of the right quantity is $f(x)$. In modern terminology we take the limit of the expression as $h \to 0$:

$$\lim_{h\to 0} f(x) < \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} < \lim_{h\to 0} f(x+h).$$

In modern parlance the middle limit is the definition of the derivative and the rightmost limit follows from the continuity of the function f; and so in any case our reasoning gives:

$$f(x) \leq F'(x) \leq f(x).$$

Leibnitz would have used the infinitesimal $h = dx$ and calculated as follows:

$$dx \cdot f(x) < dF < dx \cdot f(x+dx).$$

Where dF is the infinitesimal "area" under the curve above the region between x and $x+dx$. [Note: we are doing the case for an increasing function so that $f(x) < f(x+dx)$.] After dividing by dx, this becomes:

$$f(x) \leq \frac{dF}{dx} \leq f(x+dx).$$

Then using the "properties of infinitesimals," the dx disappears from the $f(x+dx)$ expression and the middle term is the derivative of $F(x)$. In which case we have $F'(x) = f(x)$. So F is an anti-derivative of f and in particular would be the one with $F(0)=0$ (because the area from $x=0$ to $x=0$ is 0 and using the fact that two antiderivatives of a function only differ by a constant.) Then the area under the graph, above the x-axis and between the vertical lines through $x=a$ and $x=b$ (for $b>a$) would be:

$$\text{Area } = F(b) - F(a).$$