

## The Interpolation Problem; the Gamma Function

The interpolation problem: given a function with values on some discrete set, like the positive integer, then what would be a reasonable way to extend the function so that it is defined for all the reals? Euler wanted to do this for the factorial function. He presented his results in a 1729 paper. Euler concluded that

$$n! = \int_0^1 (-\ln x)^n dx.$$

Observe that (in modern calculation)

$$\begin{aligned} \int_0^1 -\ln x \, dx &= -x \ln x + x \Big|_0^1 \\ &= 1 + \lim_{x \rightarrow 0^+} x \ln x \\ &= 1 + \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= 1 + \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= 1 + \lim_{x \rightarrow 0^+} \frac{-x}{1} = 1 - 0 = 1. \end{aligned}$$

Which is consistent with  $1! = 1$ . Now consider the following integration,

$$\int_0^1 (-\ln x)^n dx.$$

We apply the (Leibnitz version) of the integration by parts formula  $\int u \, dv = uv - \int v \, du$ .

$$\begin{aligned} dv &= 1 & v &= x \\ u &= (-\ln x)^n & du &= n(-\ln x)^{n-1} \left(-\frac{1}{x}\right) dx \end{aligned}$$

$$\int_0^1 (-\ln x)^n dx = x(-\ln x)^n \Big|_0^1 + n \int_0^1 (-\ln x)^{n-1} dx$$

Since (in modern notation and using the same limit techniques as above)

$$\lim_{x \rightarrow 0^+} x(\ln x)^n = 0$$

(Euler would have probably said that  $0(\ln(0))^n = 0$ .) We have:

$$\int_0^1 (-\ln x)^n dx = n \int_0^1 (-\ln x)^{n-1} dx$$

which is consistent with our inductive definition of the factorial function:  $n! = n \cdot (n-1)!$ ; so for integers the formula clearly works. Then he used the transformation  $t = -\ln x$  so that  $x = e^{-t}$  to obtain the following. Note that when  $x = 1, t = 0$  and when  $x \rightarrow 0, t \rightarrow \infty$ .

$$\begin{aligned} \int_0^1 (-\ln x)^n dx &= \int_{\infty}^0 (t)^n d(e^{-t}) \\ &= \int_{\infty}^0 -(t)^n e^{-t} dt \\ &= \int_0^{\infty} (t)^n e^{-t} dt \end{aligned}$$

which is the modern formulation and leads us to the definition of the gamma function. For historical reasons (the fact that Gauss'  $\pi$  function was  $\pi(n) = (n-1)!$ ) the gamma function  $\Gamma$  is defined by

$$\begin{aligned} \Gamma(n+1) &= \int_0^{\infty} (t)^n e^{-t} dt \\ &= n! \end{aligned}$$

Lets calculate  $\Gamma(\frac{1}{2})$ :

$$\begin{aligned} \Gamma(n+1) &= \int_0^{\infty} (t)^n e^{-t} dt \\ \Gamma\left(\frac{1}{2}\right) &= \int_0^{\infty} (t)^{-\frac{1}{2}} e^{-t} dt \end{aligned}$$

consider the change of variable  $t = x^2$ :

$$\begin{aligned}
 \int_0^\infty (t)^{-\frac{1}{2}} e^{-t} dt &= \int_0^\infty (x^2)^{-\frac{1}{2}} e^{-x^2} dx^2 \\
 &= \int_0^\infty \frac{1}{x} e^{-x^2} 2x dx \\
 &= 2 \int_0^\infty e^{-x^2} dx \\
 &= \int_{-\infty}^\infty e^{-x^2} dx \\
 \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}
 \end{aligned}$$

So we need the calculation of  $\int_{-\infty}^\infty e^{-x^2} dx$ . I'll remind us how it's done. We will use a change of variable from rectangular to polar coordinates. So we will need  $|J|$  for the Jacobian.

$$\begin{aligned}
 x &= r \cos \theta \\
 y &= r \sin \theta
 \end{aligned}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix}$$

$$|J| = r \cos^2 \theta + r \sin^2 \theta = r$$

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy \\
&= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} |J| dr d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \\
&= \int_0^{2\pi} \left. -\frac{1}{2} e^{-r^2} \right|_0^{\infty} d\theta \\
&= \int_0^{2\pi} \frac{1}{2} d\theta \\
&= \frac{1}{2} \cdot 2\pi = \pi \\
\int_{-\infty}^{\infty} e^{-x^2} dx &= \sqrt{\pi}
\end{aligned}$$

From our probability and/or statistics class we will want to calculate

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Using a change of variable  $u = \frac{x-\mu}{\sqrt{2}\sigma}$  so that  $du = \frac{1}{\sqrt{2}\sigma}$  So

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx &= \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2}\sigma du \\
&= \sqrt{2}\sigma \sqrt{\pi} = \sigma \sqrt{2\pi}
\end{aligned}$$

Which gives us the normal distribution probability density function:

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

**Factorial like property of the gamma function.**

From the definition of the gamma function we have the following:

$$\begin{aligned}\Gamma(n+1) &= n! \\ \Gamma(n+2) &= (n+1)! \\ &= (n+1)\Gamma(n+1).\end{aligned}$$

So the choice of the gamma function as a generalization of the factorial would be particular nice if we could show this for all  $x$  and not just for positive integers. We'd like to prove that  $\Gamma(x) = (x-1)\Gamma(x-1)$ . In the following assume  $x$  is a particular positive real number (we'll see why we need positive below). From our definition of  $\Gamma(x)$  we have

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

We proceed with integration by parts using Leibnitz notation treating  $x$  as a constant:

$$\begin{aligned}\int u dv &= uv - \int v du \\ \Gamma(x) &= \int_0^{\infty} t^{x-1} e^{-t} dt \\ &\quad u = t^{x-1}; dv = e^{-t} dt \\ &\quad du = (x-1)t^{x-2} dt; v = -e^{-t} \\ \int_0^{\infty} t^{x-1} e^{-t} dt &= \int_0^{\infty} t^{x-1} e^{-t} dt \\ &= -t^{x-1} e^{-t} \Big|_0^{\infty} - \int_0^{\infty} (x-1)t^{x-2} (-e^{-t}) dt.\end{aligned}$$

It's a simple exercise to evaluate  $t^{x-1} e^{-t} \Big|_0^{\infty}$  as long as  $x \geq 1$ . When  $x$  is less than 1 the lower integration limit yields division by zero so the integration will only be finite for  $x \geq 1$ . In these cases we have:

$$\begin{aligned}\int_0^{\infty} t^{x-1} e^{-t} dt &= -t^{x-1} e^{-t} \Big|_0^{\infty} - \int_0^{\infty} (x-1)t^{x-2} (-e^{-t}) dt \\ &= 0 + \int_0^{\infty} (x-1)t^{x-2} e^{-t} dt \\ \Gamma(x) &= (x-1)\Gamma(x-1).\end{aligned}$$

I add one more observation by Gauss where he generalized the gamma function to a function of a complex variable  $z$ :

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{\frac{z}{k}}.$$

The number  $\gamma$  is Euler's constant. Note the use of “infinite” products.