Example of a Method of Exhaustion proof using modern notation and graphics.

The axiom of Archimedes is given as: *Given two quantities there is an integer n so that n times the first quantity can be made greater than the second quantity.*
Determine where this axiom was used in the following example.

Example: Find the area of the sector of the parabola $y = x^2$ (in modern notation) bounded by the line segment from $(1,1)$ to $(-1,1)$. This region sits inside the rectangle $[-1,1] \times [0,1]$. According to Archimedes this section of this parabola would have area $\frac{2}{3}$ the base (of length 2 here) times the height (or length 1 here); area $= \frac{2}{3} \times 2 \times 1 = \frac{4}{3}$. I will use the method of exhaustion to find the area of the complementary region from $x = 0$ to $x = 1$; according to Archimedes this should have an area of $\frac{1}{3}$.

Given:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let A denote the area under the parabola $y = x^2$ from 0 to 1. Suppose that

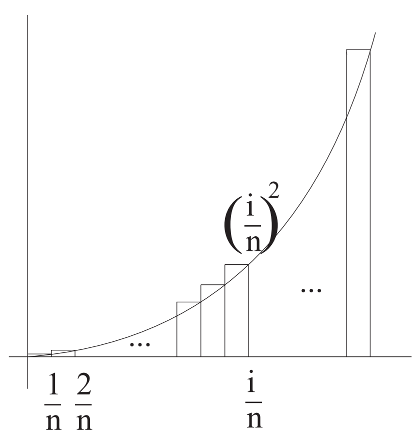

Figure 1: $y = x^2$

the area under the parabola is **not** $\frac{1}{3}$; $A \neq \frac{1}{3}$. There are two cases:

(1.) $A > \frac{1}{3}$;

(2.) $A < \frac{1}{3}$.

Suppose now that the interval from 0 to 1 is partitioned into n intervals each of length $\frac{1}{n}$ and rectangles are constructed to contain the area below the curve as illustrated in figure 1. We know from our diagram, concerning the upper sum of the rectangles, that no matter what positive integer n is chosen that(since the rectangles contain the area under consideration):

$$\sum_{i=1}^{n} \frac{i^2}{n^3} \quad > \quad A. \tag{1}$$

So:

$$\frac{1}{n^3}\sum_{i=1}^{n} i^2 \quad > \quad A \tag{2}$$

$$\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \quad > \quad A. \tag{3}$$

We consider case (1.) first. Let $\epsilon = A - \frac{1}{3}$ then $\epsilon > 0$. So there is an integer n so that

$$n \quad > \quad \frac{1}{3\epsilon}. \tag{4}$$

Therefore, for any such n:

$$\begin{aligned} n &> \frac{1}{3\epsilon} \\ n &> \frac{1}{\epsilon} \\ \frac{1}{n} &< \epsilon \\ \frac{1}{2n} &< \frac{\epsilon}{2} \end{aligned}$$

and:

$$\begin{aligned} n &> \frac{1}{3\epsilon} \\ n^2 &> \frac{1}{3\epsilon} \\ 3n^2 &> \frac{1}{\epsilon} \\ \frac{1}{3n^2} &< \epsilon \\ \frac{1}{6n^2} &< \frac{\epsilon}{2}. \end{aligned}$$

Now combining these two we get:

$$\begin{aligned} \frac{1}{2n} + \frac{1}{6n^2} &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ \frac{3n+1}{6n^2} &< \epsilon \\ \frac{3n^2+n}{6n^3} &< \epsilon \\ \frac{2n^3+3n^2+n-2n^3}{6n^3} &< \epsilon \\ \frac{2n^3+3n^2+n}{6n^3} - \frac{2n^3}{6n^3} &< \epsilon \\ \frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{3} &< \epsilon \\ \frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{3} &< \epsilon. \end{aligned}$$

But from equation (1) and the fact that $A - \frac{1}{3} = \epsilon$, the previous equation implies:

$$A - \frac{1}{3} < \frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{3} < A - \frac{1}{3}$$

which is impossible. So our assumption that $A > \frac{1}{3}$ is false. The argument for the case that $A < \frac{1}{3}$ is very similar (except that we use rectangle below the curve).

Solution. By the symmetry of the parabola, the area of the desired sector will be the area of the rectangle $[-1, 1] \times [0, 1]$ which will be 2 minus twice the area of the region I found; this will be $2 \cdot 1 - 2 \cdot \frac{1}{3} = \frac{4}{3}$ which is two thirds the base times the height which is the formula obtained by Archimedes.

Scratch Work:

How did I know what to do? Here's my scratch work. To get a contradiction, for the selected ϵ I want an integer n so that:

$$\sum_{i=1}^{n} \frac{i^2}{n^3} - \frac{1}{3} \quad < \quad \epsilon.$$

So using the formula, I want an integer n so that:

$$\sum_{i=1}^{n} \frac{i^2}{n^3} \quad = \quad \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6};$$

so I want the following

$$\frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{3} \quad < \quad \epsilon.$$

Which yields:

$$\begin{aligned} \frac{2n^3+3n^2+n}{6n^3} - \frac{1}{3} &< \epsilon \\ \frac{2n^3+3n^2+n-2n^3}{6n^3} &< \epsilon \\ \frac{3n^2+n}{6n^3} &< \epsilon \\ \frac{3n^2}{6n^3} + \frac{n}{6n^3} &< \epsilon \\ \frac{1}{2n} + \frac{1}{6n^2} &< \epsilon. \end{aligned}$$

Now trying to solve this last line for n is messy and involves the quadratic equation and the consideration of two roots, but in order for this to hold it would be sufficient if each of the terms on the left are less than $\frac{\epsilon}{2}$. So toward that end we'll solve these two equations separately:

$$\frac{1}{2n} \quad < \quad \frac{\epsilon}{2} \tag{5}$$

$$\frac{1}{6n^2} \quad < \quad \frac{\epsilon}{2}. \tag{6}$$

First we'll solve equation (3) for n:

$$\begin{aligned} \frac{1}{2n} &< \frac{\epsilon}{2} \\ \frac{1}{n} &< \epsilon \\ n &> \frac{1}{\epsilon}. \qquad (7) \end{aligned}$$

Next we'll solve equation (4) for n:

$$\begin{aligned} \frac{1}{6n^2} &< \frac{\epsilon}{2} \\ \frac{1}{n^2} &< 3\epsilon \\ n^2 &> \frac{1}{3\epsilon} \qquad (8) \\ n &> \frac{1}{3\epsilon}. \qquad (9) \end{aligned}$$

Now the step from (8) to (9) is not correct in the sense that (9) does not follow from (8); but this is "scratch work" and ultimately I will want to do these steps backwards. So I have to think "backwards" and step (8) follows from step (9) which is what I'll need for my proof. Finally notice that step (7) follows from step (9). Therefore it is sufficient for me to use equation (9) for the condition that I need. So that's how I got the condition (2) above that I needed for the proof and the proof itself consists in doing these "backward thinking" steps forward.