Finding Series Expansions for functions

Suppose that you want to express a function $f(x)$ as an infinite series. Maclaurin did the following (1742); he assumed that there was a power series expansion for a function $f(x)$ as expressed below and then solved for the unknown constants $A, B, C, D, \ldots$:

$$\begin{aligned}
f(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \\
f'(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \\
f''(x) &= 2C + 2 \cdot 3Dx + 3 \cdot 4Ex^2 + \ldots \\
f'''(x) &= 2 \cdot 3D + 2 \cdot 3 \cdot 4Ex + \ldots \\
\vdots & \quad \vdots
\end{aligned}$$

Setting $x = 0$:

$$\begin{aligned}
f(0) &= A \\
f'(0) &= B \\
f''(0) &= 2C \\
f'''(0) &= 3!D \\
\vdots & \quad \vdots \\
f^{(n)}(0) &= n!Z \\
\vdots & \quad \vdots.
\end{aligned}$$

So:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{3!}x^4 + \ldots + \frac{f^{(n)}(0)}{n!}x^n \ldots$$

Once Maclaurin had this result he proceeded to use it without worrying about convergence.

Trigonometric series.

The use of these series also came out of the interpolations problem. We're interested in representing a function f in the following form.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \Big(A_n \sin(nx) + B_n \cos(nx) \Big)$$

The differentiation trick will not work to help us evaluate the coefficients of these series. So we use an integration technique. First we need to know how to integrate functions like $\sin(mx)\sin(nx)$. To do these we recall some of our trigonometry identities.

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)\\ \cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\cos(\beta)\end{aligned}$$

From these we can get the following multiplication to addition conversion formulas:

$$\begin{aligned}\sin(\alpha)\cos(\beta) &= \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))\\ \cos(\alpha)\cos(\beta) &= \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))\\ \sin(\alpha)\sin(\beta) &= \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))\end{aligned}$$

Therefore for $m \neq n$ since the integral of the sine or cosine over an integer number of periods is zero we have:

$$\begin{aligned}\int_0^{2\pi} \sin(mx)\cos(nx)dx &= \frac{1}{2}\int_0^{2\pi} \sin((m+n)x) - \sin((m-n)x)dx = 0\\ \int_0^{2\pi} \cos(mx)\cos(nx)dx &= \frac{1}{2}\int_0^{2\pi} \cos((m+n)x) + \cos((m-n)x))dx = 0\\ \int_0^{2\pi} \sin(mx)\sin(nx)dx &= \frac{1}{2}\int_0^{2\pi} (\cos((m-n)x) - \cos((m+n)x)dx = 0.\end{aligned}$$

And for $m = n$:

$$\begin{aligned}\int_0^{2\pi} \sin(mx)\cos(mx)dx &= \frac{1}{2}\int_0^{2\pi} \sin(2mx) - 0dx = 0\\ \int_0^{2\pi} \cos^2(mx)dx &= \frac{1}{2}\int_0^{2\pi} 1 + \cos(2mx)dx = \frac{1}{2}2\pi = \pi\\ \int_0^{2\pi} \sin^2(mx)dx &= \frac{1}{2}\int_0^{2\pi} (1 - \cos(2x))dx = \frac{1}{2}2\pi = \pi.\end{aligned}$$

Suppose now that

$$f(x) = A_0 + \sum_{n=1}^{\infty}\Big(A_n \sin(nx) + B_n \cos(nx)\Big)$$

Then

$$
\begin{aligned}
\int_0^{2\pi} f(x)\sin(mx)dx &= \int_0^{2\pi} A_0\sin(mx) + \sum_{n=1}^{\infty}\Big(A_n\sin(nx)\sin(mx) + B_n\cos(nx)\sin(mx)\Big)dx \\
&= 0 + A_m\pi + 0 \\
A_m &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)dx \\
\int_0^{2\pi} f(x)\cos(mx)dx &= \int_0^{2\pi} A_0\cos(mx) + \sum_{n=1}^{\infty}\Big(A_n\sin(nx)\cos(mx) + B_n\cos(nx)\cos(mx)\Big)dx \\
&= 0 + B_m\pi + 0 \\
B_m &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)dx \\
\int_0^{2\pi} f(x)dx &= \int_0^{2\pi} A_0 + \sum_{n=1}^{\infty}\Big(A_n\sin(nx) + B_n\cos(nx)\Big)dx \\
&= \int_0^{2\pi} A_0 dx + 0 + 0 \\
&= A_0\cdot 2\pi \\
A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx.
\end{aligned}
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