The Early Calculus.

Barrow

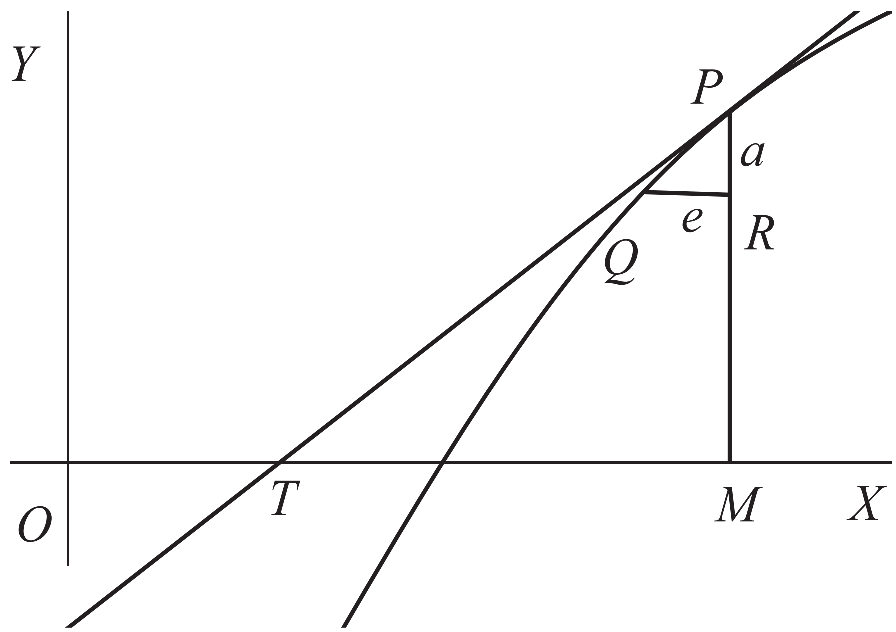

Figure 1: Barrow's Characteristic Triangle

Barrow used the following reasoning to calculate the slope of the line tangent to a curve. Consider the above figure, he assumed that the "characteristic triangle" $\triangle PQR$ is similar to the triangle made by the tangent line: $\triangle PQR \approx \triangle PTM$. Let e and a be the lengths of QR and PR respectively; then we have $P = (x, y)$ and $Q = (x - e, y - a)$ and he assumed that the point Q lies on the curve. From this assumption he calculates the ratio $\frac{a}{e}$. An example will be helpful. Let's calculate the slope of the line tangent to the circle $x^2 + y^2 = r^2$ at the point $P = (x, y)$. By assumption the point

$Q = (x - e, y - a)$ is on the curve and so satisfies the equation of the circle:

$$\begin{aligned}
(x-e)^2 + (y-a)^2 &= r^2 \\
x^2 - 2xe + e^2 + y^2 - 2ya + a^2 &= r^2 \\
-2xe + e^2 - 2ya + a^2 &= 0 \qquad (1) \\
-2xe - 2ya &= 0 \qquad (2) \\
\frac{a}{e} &= -\frac{x}{y}.
\end{aligned}$$

Step (1) follows from the fact that (x, y) satisfies the equation and step (2) is obtained by ignoring the higher powers of e and a.

We can check this using the rules of implicit differentiation learned in calculus. We want the value of y' at the point (x, y):

$$\begin{aligned}
& x^2 + y^2 = r^2 \\
\text{differentiating} \quad & 2x + 2y \cdot y' = 0 \\
& y' = -\frac{x}{y}.
\end{aligned}$$

Barrow led an interesting life: rebellious when young, went to Trinity college, was the first occupant of the Lucasin chair at Cambridge, was driven out of Cambridge during the Puritan purge (during the time of Oliver Cromwell), went on a tour of Eastern Europe that included fighting off pirates, returned to England, acquired Isaac Newton as a student, and recommended him for the Lucasin chair when he resigned the chair to pursue a career in theology, became a Doctor of Divinity (by Royal Decree) and Master of Trinity College, a position he held till the end of his life.

Newton

Some significant events in Newton's life:

• He was born Christmas day 1642 nearly a year after Galileo died (Jan. 8, 1942) - though most textbooks emphasize the fact that he was born the year that Galileo died.

• He was born after his father had died.

• During the London Plague Years 1665 - 1666 (the mortality rate in London was about 25%) he went back to the "family farm" where he made many of his mathematical and scientific discoveries. Most noteworthy of that time was his work on the splitting of white light by a prism.

• His *Methodis Serierium et Flusionum* - Methods on [Infinite] Series and Fluxions was written in 1671.

• He published the first edition of *Philosophiæ Naturalis Principia Mathematica* in Latin, generally referred to as the "Principia", in 1687.

• Master of the Mint.

• Theological research.

An example of Newton's use of infinitesimals (he used the expression "Ultimate Ratios"). He was interested in calculating the area under the curve $y = ax^r$. He used his generalized binomial expansion, which for non integer values, is an infinite series. So he assumed $r = \frac{m}{n}$ and used the fact that the derivative of the area function is the original function. So he assumed $A(x)$ (which we would write as $\int_0^x at^{\frac{m}{n}}dt$) is

$$A(x) = \frac{na}{m+n}x^{\frac{m+n}{n}}.$$

Next he proved that the function worked by showing that its derivative is the original function, he uses the letter 'o' to indicate an infinitesimal:

$$\begin{aligned}
A(y+o) &= \frac{na}{m+n}(x+o)^{\frac{m+n}{n}} \\
&= \frac{na}{m+n}x^{\frac{m+n}{n}} + \frac{m+n}{n}\frac{na}{m+n}x^{\frac{m+n}{n}}o + R\cdot o^2 \qquad (3) \\
&\quad +\text{something}\cdot\text{higher order terms of } o \\
A(y+o) - A(y) &= \frac{m+n}{n}\frac{na}{m+n}x^{\frac{m+n}{n}}o + R\cdot o^2 + S\cdot o^{3+} \\
\frac{A(y+o) - A(y)}{o} &= \frac{m+n}{n}\frac{na}{m+n}x^{\frac{m+n}{n}} + R\cdot o + S\cdot o^{2+} \\
&= \frac{m+n}{n}\frac{na}{m+n}x^{\frac{m+n}{n}} \\
&= ax^{\frac{m+n}{n}}.
\end{aligned}$$

Line (3) follows from his generalized binomial theorem.

One of the important problems that Newton had to address with his Gravitation Theory is to prove that for two spheres of constant density that to obtain the force between them he may assume that all the mass is concentrated at the center. His assumption was that for two point masses of m and M and distance r between them, that the force (as expressed with the modern gravitational constant) is:

$$F = \frac{GmM}{r^2}.$$

The solution to this problem involves two steps where each step requires

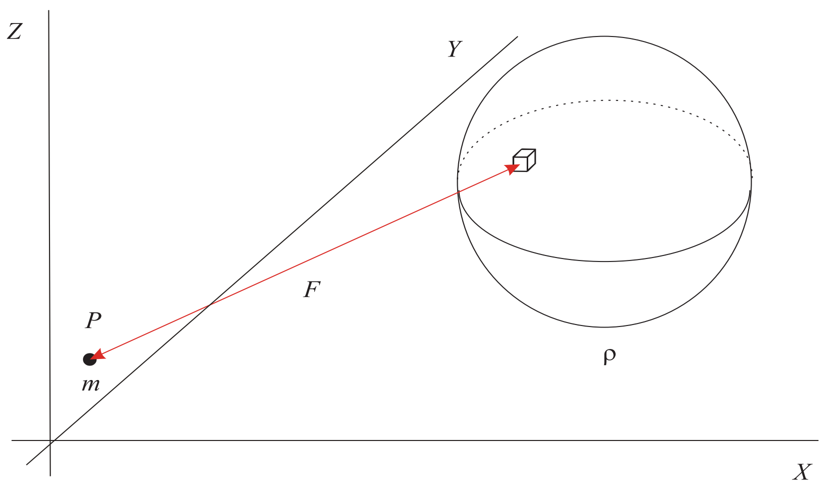

Figure 2: Force between a point mass and a sphere

the calculation of a triple integral. Using modern notation, in the pictures one can assume the dimensions of the little box is either $\Delta x \times \Delta y \times \Delta z$ or $dx \times dy \times dz$, where the modern definitions of continuity and limits is used with the former to obtain an integral that is mathematically sound and where the latter uses infinitesimals (with Liebniz's notation) which is not mathematically sound but is more akin to the techniques of Newton's time and still (miraculously?) produces the same result. If ρ is the density of the sphere and $r(x, y, z)$ is the distance from the point P to the little box, the force would look something like the following:

$$F = \int\int\int_S \frac{Gm\rho}{r(x, y, z)^2}.$$

The region inside the sphere over which the integration is performed is denoted by S. He must show that this integral is equal to

$$F = \frac{Gm\rho V}{r(x_0, y_0, z_0)^2}$$

where V is the volume of the sphere and (x_0, y_0, z_0) is the center of the sphere. The next step actually follows easily from the first step; it is to integrate over

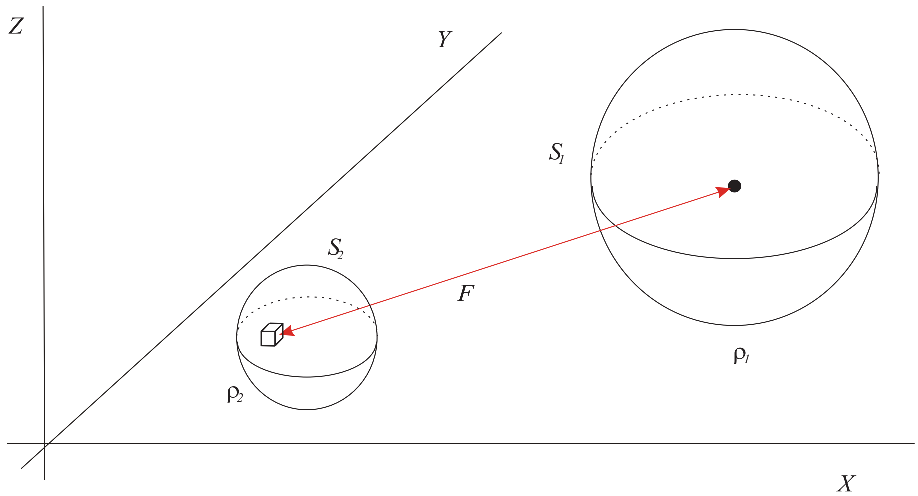

Figure 3: Force between two spheres

all the points in one sphere using $r(x, y, z)$ to be the distance from the little box in the second sphere to the center of the first sphere; and one will obtain an integral similar to the previous one. The resulting force, Newton proved, is

$$F = \frac{G\rho_1 V_1 \rho_2 V_2}{r^2}$$

where V_1 and V_2 are the volumes of the two spheres with densities ρ_1 and ρ_2 respectively and r is the distance between their centers.

Newton would have realized that the density need not be constant, it is sufficient that if varies symmetrically according to the distance from the center. One last interesting fact about the inverse square force law, is that this reduction of the problem to using the distance between centers does not work for other exponents besides 2.

Leibniz

Some significant events in Leibniz's life.

• Was also a philosopher, still being studied today. Had a theory that the universe was made up of 'Monads' each one of which reflects (in some mysterious way) the rest of the universe.

• His main paid occupation was as Historian to the House of Hanover. One of his later employers went to England to became George I, King of England (but left Leibniz behind).

• Had a theological theory that God created this world as the "Best of all Possible World" a position satirized by Voltaire in *Candid*.

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Next an example from 1676 of Leibniz's derivation of the product rule:

$$\begin{aligned} dxy &= (x+dx)(y+dy) - xy \\ &= xy + ydx + xdy + dxdy - xy \\ &= ydx + xdy + \underbrace{dxdy}_{\to 0} \\ &= ydx + xdy. \end{aligned}$$

Dividing by dt gives us:

$$\frac{dxy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}.$$

This is exactly the product rule. Now he used this to get the quotient rule:

Let $z = \frac{x}{y}$:

$$\begin{aligned} z &= \frac{x}{y} \\ x &= zy \\ dx &= d(zy) = zdy + ydz \\ ydz &= dx - zdy \\ dz &= \frac{dx - zdy}{y} \\ &= \frac{dx - \frac{x}{y}dy}{y} \\ &= \frac{ydx - xdy}{y^2}. \end{aligned}$$

Which is the quotient rule. Once we've got these two results the following is an corollary (Leibniz, 1676) for any rational number n:

$$d(x^n) = nx^{n-1}dx.$$

Leibniz said that he got the idea from Pascal that is equivalent to Barrow's "characteristic triangle."

Newton said, upon reading the work of Leibniz, "not a single previously unsolved problem was solved."

Rond d'Alembert in the *Encyclopédie* of 1751 had a discussion of a concept of the limit

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y}.$$

But it was not much less confusing than the then current definition of $\frac{dy}{dx}$ or of $\dot{y} = \dot{f}$.