Set Theory.

Undefined terms: Sets, elements of, $\in, \cap, \cup, \subset, \varnothing, \{\dots\}, \mathbb{U}$.

Axiom 0. If each of A and B is a set then

$$\begin{aligned}(A = B) &\Leftrightarrow ((\text{if } x \in A \text{ then } x \in B) \text{ and } (\text{if } x \in B \text{ then } x \in A)).\\ (A \subset B) &\Leftrightarrow (\text{if } x \in A \text{ then } x \in B).\end{aligned}$$

Axiom 1. Rules of the set operators. If each of A and B is a set then:

$$\begin{aligned}A \cap B &= \{x | x \in A \text{ and } x \in B\}\\ A \cup B &= \{x | x \in A \text{ or } x \in B\}\end{aligned}$$

Axiom 2. If x is an element and each of A and B is a set then:

$$\begin{aligned}x &\in \mathbb{U}\\ A \cap B &\subset \mathbb{U}\\ A \cup B &\subset \mathbb{U}\\ \varnothing &\subset \mathbb{U}\\ \varnothing &\subset A\\ A &\subset \mathbb{U}\end{aligned}$$

Note: $\mathbb{U}$ is sometimes called the "Universe" under consideration.

Axiom 3. If S is a set then $S \notin S$.

Theorem 5.1. The set operators $\cap$ and $\cup$ are commutative. If each of A and B is a set then:

$$\begin{aligned}A \cup B &= B \cup A\\ A \cap B &= B \cap A\end{aligned}$$

Theorem 5.2. The set operators $\cap$ and $\cup$ are associative. If each of A, B and C is a set then:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C \\ A \cup (B \cup C) &= (A \cup B) \cup C \end{aligned}$$

Theorem 5.3. Each of the set operators $\cap$ and $\cup$ distributes over the other. If each of A, B and C is a set then:

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

Definition. Let $\mathbb{U}$ be a set of elements under consideration (the Universe.) If $A \subset \mathbb{U}$ is a set then the complement of A (with respect to $\mathbb{U}$) is denoted by A^c and is the set $A^c = \{x \in \mathbb{U} | x \notin A\}$.

Thus we can assume that:

$$\begin{aligned} \varnothing^c &= \mathbb{U} \\ \mathbb{U}^c &= \varnothing. \end{aligned}$$

Theorem 5.4. If A is a set then

$$(A^c)^c = A$$

Theorem 5.5. [De Moivre's theorem for sets.]. If each of A and B is a set then:

$$\begin{aligned} (A \cup B)^c &= (A^c) \cap (B^c) \\ (A \cap B)^c &= (A^c) \cup (B^c) \end{aligned}$$

Big Exercise: Go to the class website and pull up the file on the logic (AMNotes01BLogicExercises.pdf). Then for exercise 5.1 do the following symbol replacement in exercise 1 of the logic section:

replace	with	symbol
$\wedge$	with	$\cap$
$\vee$	with	$\cup$
$\sim A$	with	(A^c)

assume that the symbols P, Q and R now are sets; then determine which are theorems and provide proofs. (You will need to figure out for yourself how to replace the symbols $\Leftarrow$ and $\Rightarrow$.)

For exercise 5.2 do the same replacements in exercise 2 of the logic section and then determine which sets yield the universe $\mathbb{U}$. In other words for 2a, prove that $P \cup (P^c) = \mathbb{U}$.

Hint: use Venn diagrams for your scratch work.

Definition. If S is a set then the *power set* of S is the set of all subsets of S; the power set is often denoted by $\mathcal{P}(S)$ or 2^S. (For the reason behind the second notation look at the next exercise.) Thus, $\mathcal{P}(S) = \{A | A \subset S\}$.

Definition. If F is a finite set then $|F|$ denotes the number of elements of the set, in other words the *cardinality* of the set.

Thus, for example $|\{1, 2, 4, 8\}| = 4$.

Exercise 5.3. Suppose that S is a finite set. Then:

(a) calculate $|\varnothing|$ and $|\mathcal{P}(\varnothing\}|$;
(b) calculate $|\{1\}|$ and $|\mathcal{P}(\{1\}|$;
(c) calculate $|\{1, 2\}|$ and $|\mathcal{P}(\{1, 2\}|$;
(d) calculate $|\{1, 2, 3\}|$ and $|\mathcal{P}(\{1, 2, 3\}|$.

Exercise 5.4. Find the pattern in exercise 5.3 and formulate and prove a theorem that says: If S is a finite set and $|S| = n$ then $|\mathcal{P}(S)| =?$. Hint: use induction.