

## Logic: Propositional Calculus

Undefined terms: Statements and statement variables; a set of logical values  $\{T, F\}$ ; operators  $\vee, \wedge, \sim$ .

Let  $\mathcal{S}$  denote the collection of statements and assume in the following that each of  $P, Q$  and  $R$  is a statement.

Axiom L0. To any statement  $P$  a logical value can be assigned. Two sentences are logically equivalent if they have the same truth values for the same truth values of their clauses.

Definition. The statement  $P$  is said to be true if and only if it has truth value  $T$ ; the statement  $P$  is said to be false if and only if it has truth value  $F$ .

Interpretations: The statement  $P \vee Q$  is interpreted to mean “ $P$  or  $Q$ ”; the statement  $P \wedge Q$  is interpreted to mean “ $P$  and  $Q$ ”; and the statement  $\sim P$  is interpreted to mean “not  $P$ ”. These interpretations should be consistent with your understanding of the grammar of our (English in our case) language.

Axiom L1.

$P$	$\sim P$
$T$	$F$
$F$	$T$

Axiom L2.

$P$	$Q$	$P \vee Q$
$T$	$T$	$T$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

Axiom L3.

$P$	$Q$	$P \wedge Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$F$

Theorem 1.1. If  $P$  is a statement then:

- i.  $P \vee P = P$
- ii.  $P \wedge P = P$
- iii.  $\sim(\sim P) = P$

Theorem 1.2. The operators  $\vee$  and  $\wedge$  are commutative: If each of  $P$  and  $Q$  is a statement then:

- i.  $P \vee Q = Q \vee P$
- ii.  $P \wedge Q = Q \wedge P$

Theorem 1.3. The operators  $\vee$  and  $\wedge$  are associative: If each of  $P, Q$  and  $R$  is a statement then:

- i.  $P \vee (Q \vee R) = (P \vee Q) \vee R$
- ii.  $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$

Theorem 1.4. Each of the operators  $\vee$  and  $\wedge$  distributes over the other: If each of  $P, Q$  and  $R$  is a statement then:

- i.  $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
- ii.  $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

Theorem 1.5. [De Morgan's Law for logic.] If each of  $P$  and  $Q$  is a statement then:

- i.  $\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$
- ii.  $\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$

Definition. If each of  $P$  and  $Q$  is a statement then the statement  $P \Rightarrow Q$  [read " $P$  implies  $Q$ "] has the following truth values.

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

Theorem 1.6. If each of  $P$  and  $Q$  is a statement then the statement  $P \Rightarrow Q$  is equivalent to the statement  $(\sim P) \vee Q$ .