Relations.

Definition. Suppose that each of A and B is a set. Then the Cartesian product $A \times B$ is defined to be:

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

Definition. The set R is a relation from A to B means that $R \subset A \times B$. If $(a, b) \in R$ then "a is related to b" is often denoted by aRb.

Definition. If R is a relation from the set A to the set B then the domain (Dom) and range (Rng) of R are defined as the following:

$$\begin{aligned} \text{Dom}(R) &= \{a \in A | \text{ there exists a } b \in B \text{ such that } (a, b) \in R\} \\ \text{Rng}(R) &= \{b \in B | \text{ there exists an } a \in A \text{ such that } (a, b) \in R\}. \end{aligned}$$

Definition. A relation f from the set A to the set B is said to be a function if for each $x \in \text{Dom}(f)$ there is a unique element $y \in B$ so that $(x, y) \in f$. Notation. If f is a function and $(x, y) \in f$ then the unique element y is denoted by $f(x)$.

Definition. If R is a relation from the set A to the set B then the inverse relation, written as R^{-1}, is a relation from the set B to the set A defined by:

$$R^{-1} = \{(b, a) | (a, b) \in A\}.$$

Definition. If R is a relation from the set A to the set B and S is a relation from the set B to the set C then the composition of S and R relations, written as $S \circ R$, is a relation from the set A to the set C defined by:

$$S \circ R = \{(a, c) | \text{ there exists } b \in B \text{ so that } (a, b) \in R, (b, c) \in S\}.$$

Example 6.1. Let $R = \{(n, m) \Big| |n - 3| + |m - 5| = 20; n, m \in \mathbb{Z}\}$. Then R is a relation.

a.) Find the domain and range of R.

b.) Find R^{-1}.
c.) Find $(R^{-1})^{-1}$.

Example 6.2. Let $R = \{(n, m) | n < m; n, m \in \mathbb{Z}\}$. Then R is a relation.
a.) Find the domain and range of R.
b.) Find R^{-1}.

Following are some helpful properties of relations, you may already have seen some of these in working with functions. They generalize properties of functions. You may wish to prove them:

Property 6.1. Suppose that $f : A \to B$ is a function and $g = f^{-1}$ is also a function then:

$$\begin{aligned} g(f(x)) &= x \quad \text{for each } x \in \mathrm{Dom}(f) \\ f(g(y)) &= y \quad \text{for each } y \in \mathrm{Rng}(f). \end{aligned}$$

Question. Is it necessary that g be a function for the theorem to hold?

Property 6.2. If R is a relation from the set A to the set B then

$$(R^{-1})^{-1} = R.$$

Property 6.3. If f is a function from the set A to the set B and g is a function from the set B to the set C, then

$$(g \circ f)^{-1} = (f^{-1}) \circ (g^{-1}).$$

Examples.
(i) Give an example to show that $f \circ g \neq g \circ f$. (ii) Give an example of a function whose inverse is not a function.

Equivalence Relations.

Definitions. Suppose that R is a relation from the set A to itself. We will use the notation $a \sim b$ to mean that a and b are in A and a is related to b or equivalently $(a, b) \in R$. Then:

R is said to be *reflexive* if $x \sim x$ for all $x \in A$.
R is said to be *symmetric* if it is true that if $x \sim y$ then $y \sim x$.
R is said to be *transitive* if it is true that if $x \sim y$ and $y \sim z$ then $x \sim z$.

Definition. A relation from a set into itself is said to be an *equivalence relation* if it is reflexive, symmetric and transitive.

Definition. Let $R = \{(a,b) | a \sim_R b\}$ be an equivalence relation on the set A. Then for each $x \in A$, we define $[x]_R = \{y | y \sim_R x\}$; this is called the equivalence class of x. If the relation is understood from the context, then the subscript may be omitted.

Exercise 6.1. Let $A = \mathbb{Z}$ and n be a positive integer; let R be the relation so that $x \sim y$ if and only if $n|(y-x)$. Show that R is an equivalence relation.

(Notation. We use the notation $y = x \mod (n)$ to indicate this specific relation R. Another notation is $x \equiv_n y$. If the integer n is understood then the notation $x \equiv y$ may be used.)

Notation: $\mathbb{Z}_n = \{[m]_{\equiv_n} | m \in \mathbb{Z}\}$.

Calculate $|\mathbb{Z}_n|$. [Hint: Do it for 2, 3, 4, ... first, then generalize.]

Theorem 6.4. Let R be an equivalence relation on the set A. Then the following are equivalent.

1. $[x] \cap [y] \neq \varnothing$;
2. $x \sim y$;
3. $[x] = [y]$.

Exercise 6.2.

a. Let $A = \mathbb{R}$ and $\mathbb{Q}$ be the rational numbers; let R be the relation so that $x \sim y$ if and only if $(y-x) \in \mathbb{Q}$. Show that R is an equivalence relation. Determine $[\sqrt{2}]_R$; which of the following numbers are in $[\sqrt{2}]$: $\frac{1}{2}$, $\sqrt{8}$, $\sqrt{3}$.

b. Let $B = \mathbb{R}^+$ and $\mathbb{Q}$ be the rational numbers; let S be the relation on B so that $x \sim_S y$ if and only if $\frac{x}{y} \in \mathbb{Q}$. Show that S is an equivalence relation. Determine $[\sqrt{2}]_S$. Again determine which of the following numbers are in $[\sqrt{2}]_S$: $\frac{1}{2}$, $\sqrt{8}$, $\sqrt{3}$

Exercise 6.3. Let $A = \mathbb{Z} \times \{\mathbb{Z} - \{0\}\}$; let $\equiv$ be the relation so that $(a,b) \equiv (c,d)$ if and only if $ad = cb$. Show that $\equiv$ is an equivalence relation.

Determine $[(1,1)], [(2,3)], [(-3,5)]$ (give a formula if you can).

Exercise 6.4. Let $\mathbb{Z}_n = \{[x]_{\equiv_n} | x \in \mathbb{Z}\}$. Determine the cardinality $|\mathbb{Z}_n|$ of $\mathbb{Z}_n$.

Definition. Suppose that S is a set and Γ an index set (often Γ will be the positive integers); then the collection of sets $\{S_\gamma\}_{\gamma\in\Gamma}$ is called a *partition* of S if and only if:
(i) $S = \cup_{\gamma\in\Gamma} S_\gamma$;
(ii) $S_\gamma \cap S_\delta = \varnothing$ whenever $\gamma \neq \delta$;
(iii) $S_\gamma \neq \varnothing$ for all $\gamma \in \Gamma$.

Theorem 6.5. Suppose S is a set and $\{S_\gamma\}_{\gamma\in\Gamma}$ is a partition of S and the relation R on S defined by $x \sim y$ if and only if $\{x, y\} \subset S_\gamma$ for some $\gamma \in \Gamma$. Then R is an equivalence relation on S.

Suppose S is a set and $\equiv$ is an equivalence relation on S, then $\{[x]_\equiv \mid x \in S\}$ is a partition of S.

Exercise 6.6. Let S denote the set of all finite subsets of $\mathbb{R}$ and let $A \sim B$ mean that $|A| = |B|$. Show that $\sim$ is an equivalence relation on S.

Definitions: Suppose A and B are sets and $f : A \to B$ is a function.
The function f is said to be *onto* (or *surjective*) if for each $b \in B$ there is an $a \in A$ so that $f(a) = b$.
The function f is said to be *one-to-one* (or *injective*) if whenever $f(x) = f(y)$ we have $x = y$.

Definition. Suppose that $\equiv_A$ is an equivalence relation on the set A, $\equiv_B$ is an equivalence relation on the set B; $\mathcal{A}$ and $\mathcal{B}$ are the sets of equivalence classes:

$$\begin{aligned} \mathcal{A} &= \{[a] | a \in A\} \\ \mathcal{B} &= \{[b] | b \in B\} \end{aligned}$$

Suppose further that $f : A \to B$ is a function. Then $F : \mathcal{A} \to \mathcal{B}$ defined by

$$F([x]_{\equiv_A}) = [f(x)]_{\equiv_B}$$

is *well-defined* means that whenever $x \equiv_A y$ we have $f(x) \equiv_B f(y)$.

Exercise 6.7. Determine which of the following are well defined functions:

a. $F : \mathbb{Z}_5 \to \mathbb{Z}_5$ where $F([x]_5) = [2x + 1]_5$.
b. $F : \mathbb{Z}_5 \to \mathbb{Z}_5$ where $F([x]_5) = [x^2]_5$.
c. $F : \mathbb{Z}_5 \to \mathbb{Z}_4$ where $F([x]_5) = [2x + 1]_4$.
d. $F : \mathbb{Z}_3 \to \mathbb{Z}_6$ where $F([x]_3) = [2x + 1]_6$.
e. $F : \mathbb{Z}_3 \to \mathbb{Z}_6$ where $F([x]_3) = [5x + 3]_6$.
f. $F : \mathbb{Z}_3 \to \mathbb{Z}_6$ where $F([x]_3) = [2x^2 + 7]_6$.
g. $F : \mathbb{Z}_3 \to \mathbb{Z}_6$ where $F([x]_3) = [x^2]_6$.

Exercise 6.8. For each of a-g of exercise, for the ones that are well defined determine if the function is one-to-one or onto. Then for the following functions, determine if they are well defined, if so determine if they are one-to-one or onto.

h. $F : \mathbb{Z}_{31} \to \mathbb{Z}_{31}$ where $F([x]_{31}) = [x + 16]_{31}$.

i. $F : \mathbb{Z}_{31} \to \mathbb{Z}_{31}$ where $F([x]_{31}) = [7x + 16]_{31}$. [Hint: $gcd(7, 31) = 1$ and values for x and y so that $31x + 7y = 1$ can be easily obtained by observing that $7 \cdot 9 = 63 = 31 \cdot 2 + 1$.]

j. $F : \mathbb{Z}_{30} \to \mathbb{Z}_{30}$ where $F([x]_{30}) = [x + 16]_{30}$.
k. $F : \mathbb{Z}_{30} \to \mathbb{Z}_{30}$ where $F([x]_{30}) = [5x + 16]_{30}$.
l. $F : \mathbb{Z}_{30} \to \mathbb{Z}_5$ where $F([x]_{30}) = [7x + 16]_5$.
m. $F : \mathbb{Z}_5 \to \mathbb{Z}_{30}$ where $F([x]_{30}) = [6x + 16]_5$.