Homomorphisms.

Definition. If $(G_1, *)$ and $(G_2, \diamond)$ are sets with operations $*$ and $\diamond$ respectively and $F : G_1 \to G_2$ is a function, then F is called a *homomorphism* if and only if

$$F(x * y) \quad = \quad F(x) \diamond F(y).$$

Definitions. Two sets are said to be homomorphic if there is a homomorphism, with regard to their respective operations, from one onto the other.

Two sets are said to be isomorphic if there is an isomorphism, with regard to their respective operations, from one onto the other.

Definition. A homomorphism that is one-to-one is called an *isomorphism.*

Exercise 7.1. Prove that the following functions are homomorphisms (Caution: I think one of them is not!); which, if any, are isomorphisms:

a. $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ where $f(x) = 5x$.
b. $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ where $f(x) = 5x$.
c. $f : (\mathbb{R}, \cdot) \to (\mathbb{R}, \cdot)$ where $f(x) = 5x$.
d. $f : (\mathbb{R}, +) \to (\mathbb{R}, \cdot)$ where $f(x) = 2^x$.
e. $f : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$ where $f(x) = \ln(x)$.
f. $f : (\mathbb{C}, +) \to (\mathbb{R}, +)$ where $f(x + yi) = 3x + 5y$ ($\mathbb{C}$ denotes the complex numbers).

Exercise 7.2. Determine if the following are homomorphisms (isomorphisms), in each case make sure to check to see if the function is well-defined:

a. $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_5, +_5)$ where $F([x]_5) = [2x + 1]_5$.
b. $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_5, +_5)$ where $F([x]_5) = [2x]_5$.
c. $F : (\mathbb{Z}_{10}, +_{10}) \to (\mathbb{Z}_5, +_5)$ where $F([x]_5) = [2x]_5$.
d. $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_{10}, +_{10})$ where $F([x]_5) = [2x]_5$.
e. $F : (\mathbb{Z}_{31}, +_{31}) \to (\mathbb{Z}_{31}, +_{31})$ where $F([x]_{31}) = [7x]_{31}$.
f. $F : (\mathbb{Z}_5, \cdot_5) \to (\mathbb{Z}_5, \cdot_5)$ where $F([x]_5) = [2x + 1]_5$.
g. $F : (\mathbb{Z}_5, \cdot_5) \to (\mathbb{Z}_5, \cdot_5)$ where $F([x]_5) = [2x]_5$.
h. $F : (\mathbb{Z}_4, +_4) \to (\mathbb{Z}_5, \cdot_5)$ where $F([x]_4) = [2]_5^x$.

Exercise 7.3. Suppose $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_n, +_n)$ where $F([x]_5) = [ax + b]_n$ is a homomorphism. Then:

1.) Either a or n is divisible by 5.
2.) $b \sim_n 0$.
3.) $F([0]_5) = [0]_n$.

Theorem 7.1 Suppose $F : (\mathbb{Z}_m, +_m) \to (\mathbb{Z}_n, +_n)$ is a homomorphism. Then $F([0]_m) = [0]_n$.