## Constructing the Reals from the Rationals

We denote the rational numbers by $\mathbb{Q}$ and from this point on we will use the standard $\frac{a}{b}$ notation; thus,

$$\frac{a}{b} \;=\; [(a,b)].$$

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is called a Cauchy sequence if for each $\epsilon > 0, \epsilon \in \mathbb{Q}$, there is an integer $N \in \mathbb{Z}^+$ so that if $k$ and $\ell$ are integers greater than $N$ then

$$|x_k - x_\ell| \;<\; \epsilon.$$

We can use the predicate calculus to express this as

$$\forall(\epsilon > 0, \epsilon \in \mathbb{Q})\exists(N \in \mathbb{Z}^+)\forall(k,\ell > N, k,\ell \in \mathbb{Z})(|x_k - x_\ell| < \epsilon).$$

If we assume that all the numbers are rational we can abbreviate this as

$$\forall(\epsilon > 0)\exists(N \in \mathbb{Z}^+)\forall(k,\ell > N)(|x_k - x_\ell| < \epsilon).$$

Although the definition of a Cauchy sequence makes sense for real numbers, since we are using the concept to construct the reals from the rationals, for these pages of notes, we assume all numbers are rational (even if I neglect to state this assumption).

Let $S$ be the set of all Cauchy sequences of rational numbers. If each of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences (of rational numbers) then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are said to be equivalent if and only for each (rational) number $\epsilon > 0$, there exists an integer $N$ so that if $k$ is an integer greater than $N$ then:

$$|x_k - y_k| \;<\; \epsilon.$$

We will denote the equivalence relation as $\sim_C$ or as just $\sim$ if no confusion is likely to arrive. But we need to prove that it is indeed an equivalence relation.

Theorem 1. The relation $\sim$ defined above is an equivalence relation.

Exercise. Let $r \in \mathbb{Q}$ and suppose that for each positive integer $n$ we have

$$\begin{aligned} x_n &= r \\ y_n &= r + \frac{1}{n}. \end{aligned}$$

Show that

$$\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty}.$$

Exercises. For each of the following, determine if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

$$\begin{aligned} \text{a.)} x_n &= 1 + \frac{1}{n+5} \\ \text{b.)} x_n &= 2 + \frac{1}{n^2} \\ \text{c.)} x_n &= 2^n \\ \text{d.)} x_n &= \frac{1}{2^n} \\ \text{e.)} x_n &= 1 + (-1)^n \\ \text{f.)} x_n &= \frac{n-1}{n+1} \end{aligned}$$

Exercises. For each of the following, determine if $\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty}$.

$$\begin{array}{lll} \text{a.)} & x_n = 1 & y_n = 1 + \frac{1}{n+5} \\ \text{b.)} & x_n = 0 & y_n = \frac{2}{3} \\ \text{c.)} & x_n = 1 & y_n = 2 + \frac{1}{n+5} \\ \text{d.)} & x_n = 1 + \frac{1}{n} & y_n = 1 - \frac{1}{n} \\ \text{e.)} & x_n = 1 & y_n = \frac{n-1}{n+1} \\ \text{f.)} & x_n = 2 & y_n = \sum_{i=0}^{n} \frac{1}{2^i}. \end{array}$$

Definition. We will denote the set of equivalence classes of the relation $\sim_C$ as $\mathbb{R}$.

Lemma (for the definition). If each of $[\{x_n\}_{n=1}^{\infty}]$ and $[\{y_n\}_{n=1}^{\infty}]$ is a Cauchy sequence, then so is $[\{x_n + y_n\}_{n=1}^{\infty}]$; moreover, if each of $[\{x_n\}_{n=1}^{\infty}]$ and $[\{y_n\}_{n=1}^{\infty}]$ is an element of $S$, then so is $[\{x_n + y_n\}_{n=1}^{\infty}]$.

Definition. We define the operation $+_C$ on elements of $\mathbb{R}$ as follows:

$$[\{x_n\}_{n=1}^{\infty}] +_C [\{y_n\}_{n=1}^{\infty}] \quad = \quad [\{x_n + y_n\}_{n=1}^{\infty}.$$

Theorem 2. The operation $+_C$ on elements of $\mathbb{R}$ is well defined.

Notation. Let $S_{\mathbb{Q}} = \{[\{q\}_{n=1}^{\infty}] | q \in \mathbb{Q}\}$.
Observation: it is easy to verify that $S_{\mathbb{Q}} \subset S$.

Theorem 3. Define $\varphi : \mathbb{Q} \to \mathbb{R}$ as follows:

$$\varphi(q) \quad = \quad [\{q\}_{n=1}^{\infty}].$$

Then $\varphi(q)$ is an isomorphism.

Definition. We define the operation $\cdot_C$ on elements of $\mathbb{R}$ as follows:

$$[\{x_n\}_{n=1}^{\infty}] \cdot_C [\{y_n\}_{n=1}^{\infty}] \quad = \quad [\{x_n \cdot y_n\}_{n=1}^{\infty}].$$

Lemma. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence then there is number $B$ (called an upper bound) so that $|x_n| < B$.

Theorem 4. The operation $\cdot_C$ on elements of $\mathbb{R}$ is well defined.