MATH 3100 Project on Constructing the Rationals.

The project is to be typed up in a LaTex file and compiled into a pdf document. As usual, send me the compiled pdf document via email by 11"59 pm of the due date with the file name beginning with your last name. Part 1 is due in about two weeks (March 29 $\pm$). I know that some of you work in study groups and I'm fine with that (in fact I'd encourage it.) It's okay to give hints to each other (what's a "hint" is an ethical question in itself, but I think you have an intuitive notion when something is an answer and not a hint): it's not okay to do another student's homework for them and it's not okay to copy someone else's solution and present it as your own. If you do work with someone, you need to indicate who is your collaborator.

Part 1.

Exercise 1. Background. There are two ways that mathematicians consider the real number line. One way is to state a sequence of axioms that define the real number line. This is what I've already done. The other way is to use the axioms of the integers and construct the number line from the integers. This project works through the first step in constructing the reals from the axioms of the integers, the construction of the rationals from the integers. A rational numbers will be constructed as an equivalence class of pairs of integers.

Let X denote the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ and define the relation $(a, b) \sim (c, d)$ by

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

As a hint and to help you understand where this comes from, although you've probably never been told in grammar school, you really have been used to thinking of rational fractions as equivalence classes of pairs of integers, you just write the pair vertically with a line between the integers, $\frac{a}{b}$. And you can tell if two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are 'equal' if $ad = cd$. So to help with your understanding, think of an equivalence class $[(a, b)]$ as the fraction $\frac{a}{b}$.

Special Caution! Since we're working with the integers, there's no division - we don't have multiplicative inverses. So in your proofs you may not use fractions or division of quantities.

Now what needs to be done is to define addition and multiplication for these equivalence classes; to show that they have all the right properties; to find the (isomorphic) copy of the integers that sits in the rationals; and to show that the rationals satisfies the axioms of the real numbers that we've used (for this project, we won't test all of them).

i.) [Constructing the rationals.] Show that $\sim$ is an equivalence relation. Let $\mathbb{Q}$ denote the set of equivalence classes:

$$\mathbb{Q} = \{[(a,b)] | a, b \in \mathbb{Z}, b \neq 0\}.$$

ii.)[Defining the multiplication.] Define the operation $\otimes$ on the equivalence classes by

$$[(a,b)] \otimes [(c,d)] \quad = \quad [(ac, bd)].$$

Show that $\otimes$ is well defined.

iii.) Show that the following definition for addition is **not** well-defined:

$$[(a,b)] \boxplus [(c,d)] \quad = \quad [(a+c, b+d)].$$

iv.) Let the operation $\oplus$ on the equivalence classes be defined by

$$[(a,b)] \oplus [(c,d)] \quad = \quad [(ad+bc, bd)].$$

Show that $\oplus$ is well defined.