MATH 3100 Project on Constructing the Rationals.
Part 2.

Due date:11:59 pm Monday April 8.

Exercise 2.

i.) [Regarding the addition.] Let the operation $\oplus$ be defined on the equivalence classes by

$$[(a, b)] \oplus [(c, d)] \;=\; [(ad + bc, bd)].$$

Show that $\mathbb{Q}$ with the operation $\oplus$ is an abelian group.

ii.)[Regarding the multiplication.] Define the operation $\otimes$ on the equivalence classes by

$$[(a, b)] \otimes [(c, d)] \;=\; [(ac, bd)].$$

Show that $\mathbb{Q} - \{\text{the additive identity of } \oplus\}$ (i.e. the set $\mathbb{Q}$ with the additive identity removed) is an abelian group with the operation $\otimes$.

iii.) [Identifying the integers inside the rationals.] Define $\varphi : \mathbb{Z} \to \mathbb{Q}$ by $\varphi(z) = [(z, 1)]$. Show that
        a.) φ is 1-to-1.
        b.) φ is a homomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{Q}, \oplus)$.
        c.) φ is a homomorphism from $(\mathbb{Z}, \cdot)$ to $(\mathbb{Q}, \otimes)$.