Cauchy Sequences

Let $\mathbb{Q}$ denote the rational numbers and assume for the moment that all quantities mentioned below are elements of $\mathbb{Q}$. [Note: this is just a formal assumption since we have not yet formally constructed the real numbers.]

Definition. Suppose that $S = \{x_i\}_{i=1}^{\infty}$ is a sequence of numbers. Then $\{x_i\}_{i=1}^{\infty}$ is said to be a Cauchy sequence provided the following hold: If $\epsilon > 0$, then there exists an integer N_ϵ so that $|x_n - x_m| < \epsilon$ for all integers n and m greater than N_ϵ.

[Note: I use the notation N_ϵ to emphasize the fact that N_ϵ depends on ϵ; but if N_ϵ is replaced with N, the definition is equivalent.]

Equivalently:

$$(\forall(\epsilon > 0))(\exists(N_\epsilon \in \mathbb{Z}))(\forall(n, m \in \mathbb{Z}(n, m > N_\epsilon)))(|x_n - x_m| < \epsilon).$$

Exercise 10.1. Express the negation of what it means for $S = \{x_i\}_{i=1}^{\infty}$ to be a Cauchy sequence without using the negation $\sim$ symbol.

Exercise 10.2. For each of the following sequences, determine if $S = \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence:

$$\begin{aligned}
\text{a.}\quad x_n &= 2 \\
\text{b.}\quad x_n &= n^2 \\
\text{c.}\quad x_n &= 3n - 10 \\
\text{d.}\quad x_n &= \frac{1}{n} \\
\text{e.}\quad x_n &= \frac{1}{7n+3} \\
\text{f.}\quad x_n &= (-1)^n \frac{1}{n} \\
\text{g.}\quad x_n &= (-1)^n \frac{1}{7n+3} \\
\text{h.}\quad x_n &= \sum_{k=1}^{n} \frac{1}{2} \\
\text{i.}\quad x_n &= \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \\
\text{j.}\quad x_n &= \sum_{k=1}^{n} \frac{1}{k^2}.
\end{aligned}$$

Definition. A sequence$\{x_n\}_{n=1}^{\infty}$ is said to be bounded if there is a number B so that $|x_n| < B$ for all n.

Theorem 10.1 If $S = \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then S is bounded.

Theorem 10.2 If each of $S_1 = \{x_n\}_{n=1}^{\infty}$ and $S_2 = \{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\{x_n + y_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Theorem 10.3 If B is a number and $S = \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\{Bx_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Definition. The sequence of rational numbers $S = \{r_i\}_{i=1}^{\infty}$ is said to converge to 0 means that if $\epsilon > 0$ then there exists an integer N_ϵ so that if $n > N_\epsilon$ then

$$|r_n| \ < \ \epsilon.$$

Theorem 10.3b. If $S = \{r_i\}_{i=1}^{\infty}$ is a sequence of rational numbers that converges to 0, the it is a Cauchy sequence.

Exercise 10.3. For the sequences of exercise 10.2, for those that are Cauchy sequences, determine if they converge to zero and prove your answer.

Let $\hat{R}$ be the collection of all Cauchy sequences of rational numbers. Define the relation $\sim$ on $\hat{R}$ by $\{x_i\}_{i=1}^{\infty} \sim \{y_i\}_{i=1}^{\infty}$ if and only if $\{x_i - y_i\}_{i=1}^{\infty}$ converges to 0.

Theorem 10.4. The relation $\sim$ defined on $\hat{R}$ is an equivalence relation.

We will denote the collection of equivalence classes in $\hat{R}$ as $\mathbb{R}$.

Before we can examine the multiplication operator, we'll need to prove the following:

Theorem 10.4b. If each of $S_1 = \{x_n\}_{n=1}^{\infty}$ and $S_2 = \{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\{x_n \cdot y_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

[Hint: use theorem 10.1.]

Definition. We define the $+$ and $\cdot$ operators on $\mathbb{R}$ as follows:

$$\begin{aligned} \left[\{x_i\}_{i=1}^{\infty}\right] + \left[\{y_i\}_{i=1}^{\infty}\right] &= \left[\{x_i + y_i\}_{i=1}^{\infty}\right] \\ \left[\{x_i\}_{i=1}^{\infty}\right] \cdot \left[\{y_i\}_{i=1}^{\infty}\right] &= \left[\{x_i \cdot y_i\}_{i=1}^{\infty}\right]. \end{aligned}$$

Theorem 10.5. The operation $+$ on $\mathbb{R}$ is well defined.

Theorem 10.6. The operation $\cdot$ on $\mathbb{R}$ is well defined.

Theorem 10.7. There is an copy $\theta(\mathbb{Q})$ of $\mathbb{Q}$ so that θ is an isomorphism with respect to $+$ and is an isomorphism of $\mathbb{Q} - \{0\}$ with respect to $\cdot$.