Math 3100, Project 02, Due Monday April 28

Definition. The sequence of rational numbers $S = \{r_i\}_{i=1}^{\infty}$ converges to 0 means that if $\epsilon > 0$ then there exists an integer N_ϵ so that if $n > N_\epsilon$ then

$$|r_n| \;<\; \epsilon.$$

Let $\hat{R}$ be the collection of all Cauchy sequences of rational numbers. Define the relation $\sim$ on $\hat{R}$ by $\{x_i\}_{i=1}^{\infty} \sim \{y_i\}_{i=1}^{\infty}$ if and only if $\{x_i - y_i\}_{i=1}^{\infty}$ converges to 0.

Denote the collection of equivalence classes in $\hat{R}$ as $\mathbb{R}$.

Definition. We define the operation $\leq$ on $\mathbb{R}$ as follows:

$$\left[\{x_i\}_{i=1}^{\infty}\right] \;\leq\; \left[\{y_i\}_{i=1}^{\infty}\right]$$

if and only if for each $\epsilon > 0$ there exists an integer N_ϵ such that if $n > N_\epsilon$ then

$$x_n - y_n \;<\; \epsilon.$$

Prove the following theorems.

Theorem 1. The relation $\leq$ is well defined.

Definition: if M is a subset of the $\mathbb{R}$ then B is an upper bound of M means that $x \leq B$ for all $x \in M$; B is said to a least upper bound of M if B is an upper bound of M and no element less than B is an upper bound of M. [Note that $x \in \mathbb{R}$ means that $x = \left[\{x_i\}_{i=1}^{\infty}\right]$. Also note that we may assume the $B \in \theta(\mathbb{Q})$.]

Theorem 1.9. [An easier version of theorem 2.] If M is a subset of $\theta(\mathbb{Q})$ and there is an upper bound of M then there is a least upper bound of M.

Theorem 2. If M is a subset of $\mathbb{R}$ and there is an upper bound of M then there is a least upper bound of M.

Homework exercises for Friday April 25. These are supposed to be helpful for working through the project. Use the definition of $\leq$ given above to show the following.

A. If $\left[\{x_i\}_{i=1}^{\infty}\right] \leq \left[\{y_i\}_{i=1}^{\infty}\right]$ then there exists sequences $\{a_i\}_{i=1}^{\infty} \in \left[\{x_i\}_{i=1}^{\infty}\right]$ and $\{b_i\}_{i=1}^{\infty} \in \left[\{y_i\}_{i=1}^{\infty}\right]$ so that $a_i \leq b_i$ for all $i \in \mathbb{N}$.

B. If $\{a_i\}_{i=1}^{\infty} \in \left[\{x_i\}_{i=1}^{\infty}\right]$ then

$$\left[\{a_i\}_{i=1}^{\infty}\right] \leq \left[\{1 + a_i\}_{i=1}^{\infty}\right].$$

C.

$$\left[\left\{\frac{100}{n}\right\}_{n=1}^{\infty}\right] \leq \left[\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}\right].$$