Project on constructing the rational numbers $\mathbb{Q}$.

Define the set $\mathbb{Q}$ of pairs of integers as follows: $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Equivalently $(x, y) \in \mathbb{Q}$ if and only if $x, y \in \mathbb{Z}$ and $y \neq 0$. We define the following relation $\sim$ on $\mathbb{Q}$

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

Part I. Note that you may only use the axioms of the integers and the theorems that we have derived from them to do the following problems. Part I is due Friday Mar. 21.

Problem 1. Prove that $\sim$ is an equivalence relation.

[Note that now we can talk about equivalence classes.] Define:

$$[(a, b)] \quad = \quad \{(c, d) | (c, d) \sim (a, b)\}.$$

Problem 2. [Note that the operations on the right in the following are ordinary multiplication and addition on the integers.]

a.) Consider the operation $\cdot_\sim$ defined by

$$[(a, b)] \cdot_\sim [(c, d)] \quad = \quad [(ac, bd)].$$

Show that $\cdot_\sim$ is well defined.

Note, this is with respect to the equivalence relation $\sim$: In other words: show that if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ then $(ac, bd) \sim (a'c', b'd')$. Equivalently, show that if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ then $[(ac, bd)] = [(a'c', b'd')]$

b.) Consider the operation $\oplus$ defined by

$$[(a, b)] \oplus [(c, d)] \quad = \quad [(a + c, b + d)].$$

Show that $\oplus$ is not well defined.

In other words: show that there exist $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ thaat $(a + c, b + d) \nsim (a' + c', b' = d')$. Equivalently, show that there exist $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ so that $[(ac, bd)] \neq [(a'c', b'd')]$.

c.) Consider the operation $+_\sim$ defined by

$$[(a,b)] +_\sim [(c,d)] \quad = \quad [(ad+bc, bd)].$$

Show that $+_\sim$ is well defined.

In other words: show that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then $(ad+bc, bd) \sim (a'd'+b'c', b'd')$. Equivalently, show that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then $[(ad+bc,bd)] = [(a'd'+b'c', b'd')]$.

[Big hint: since we are constructing the rationals from the axioms of the integers, you should be aware that we are constructing the usual set of fractions in the form $\frac{n}{m}$. Our goal is to prove from the axioms that our $\mathbb{Q}$ is equivalent to our usual fractions. So there has to be a one-to-one relationship between $\mathbb{Q}$ and our familiar fractions. Use that to help answer the problems - then use our knowledge of the integers to prove your responses to the problems posed.]

Project on the rational numbers II.

Part II. Due Monday Mar. 24.

Define $\mathbb{Q}_\sim = \{[(a,b)] | (a,b) \in \mathbb{Q}\}$.

Problem 3a. Show that there is an "additive" identity in $\mathbb{Q}_\sim$ for the operation $+_\sim$.

In other words, show that there exists $(e_1, e_2) \in \mathbb{Q}$ so that for an arbitrary $(a,b) \in \mathbb{Q}$ we have

$$[(e_1,e_2)] +_\sim [(a,b)] \quad = \quad [(a,b)].$$

Note that since (it is easy to see that) the operation is commutative, it's not necessary so show that the identity that we select is both a left identity and a right identity (and similarly below for inverses).

Problem 3b. Show that there is an additive inverse for each element of $\mathbb{Q}$.

In other words, show that for an arbitrary $(a,b) \in \mathbb{Q}$, there exists $(c,d) \in \mathbb{Q}$ so that we have

$$[(c,d)] +_\sim [(a,b)] \quad = \quad [(e_1,e_2)].$$

Problem 4a. Show that there is a "multiplicative" identity in $\mathbb{Q}_\sim$ for the operation $\cdot_\sim$.

In other words, show that there exists $(\ell_1, \ell_2) \in \mathbb{Q}$ so that if $(a, b) \in \mathbb{Q}$, we have

$$[(\ell_1, \ell_2)] \cdot_\sim [(a, b)] \;=\; [(a, b)].$$

Problem 4b. Show that there is a multiplicative inverse for each element of $\mathbb{Q}_\sim$ except for the additive identity.

Problem 4c. Show that the additive identity does not have a multiplicative inverse.

I.e.: show that $[(e_1, e_2)]$ does not have a multiplicative inverse.

Problem 5. Show that there is an isomorphic copy of the integers in $\mathbb{Q}_\sim$ which is an isomorphism with respect to both addition ($+_\sim$) and multiplication ($\cdot_\sim$) operations.

For future needs, lets call the isomorphism θ.

Note that once we's done our proofs for problem 5, then we may stop subscripting the relationships $+_\sim$ and $\cdot_\sim$ since, where appropriate, they are isomorphic to the operations on the integers.

Project on the rational numbers III.

We need to show that the classes from the equivalence relation $\sim$ on $\mathbb{Q}$ satisfy the same axioms as the integers with regard to the operations $+$ and $\cdot$ (note that I've omitted the subscript) of the operations on the equivalence classes. We will prove a selected few axioms.

Problem 6. Show that the operation $+$ is associative. [Note that by $+$, I mean $+_\sim$ and similarly with $\cdot$.]

Problem 7. Show that the distribution axiom holds:

$$[(x, y)] \cdot ([(a, b)]) + [(c, d)]) \;=\; [(x, y)] \cdot [(a, b)] + [(x, y)] \cdot [(c, d)].$$

Problem 8. Define the inequality $<_\sim$ on the equivalence classes of $\mathbb{Q}$ (i.e. on $\mathbb{Q}_\sim$) so that the following hold.

a. Define $<_\sim$ so that if $\theta : \mathbb{Z} \to \mathbb{Q}_\sim$ is the isomorphism defined in part 2 of the project, then $x < y$ if and only if $\theta(x) <_\sim \theta(y)$.

b. Prove that $<_\sim$ is well-defined.

c. Prove that the relation $<_\sim$ satisfies axioms D4 and D5.

[Hint: show that for each $(a, b) \in \mathbb{Q}$ there exist $(a', b') \in \mathbb{Q}$ so that $b' > 0$ and $(a, b) \sim (a', b')$. And use the big hint of part I.]