Exercise 55.22 g: Show that $\sqrt[3]{24}$ irrational.

Proof. We suppose that the statement is false and assume that there are a pair of integers so that

$$\sqrt[3]{24} = \frac{a}{b}.$$

Then from the fundamental theorem of arithmetic we can represent a and b as unique products of primes so that

$$\begin{aligned} a &= p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} \\ b &= q_1^{r_1} q_2^{r_2} \dots q_\ell^{r_\ell} \end{aligned}$$

and where for each $i = 1, 2, \dots, k$, p_i is a prime number with $p_i < p_{i+1}$ for $i = 1, 2, \dots, k-1$ and similarly where for each $i = 1, 2, \dots, \ell$, q_i is a prime number with $q_i < q_{i+1}$ for $i = 1, 2, \dots, k-1$, and where the exponents are unique. So in this partiular case we can assume that

$$\begin{aligned} a &= 2^n 3^m P \\ b &= 2^r 3^s Q \end{aligned}$$

where P and Q are the product of primes no one of which is divisible by 2 or by 3. Observe that

$$\begin{aligned} a^3 &= 2^{3n} 3^{3m} P^3 \\ b^3 &= 2^{3r} 3^{3s} Q^3. \end{aligned}$$

So we have

$$\begin{aligned} \sqrt[3]{24} &= \frac{a}{b} \\ \sqrt[3]{24} b &= a \\ 24 b^3 &= a^3 \\ 2^3 \cdot 3 \cdot 2^{3r} 3^{3s} Q^3 &= 2^{3n} 3^{3m} P^3 \\ 2^{3r+3} 3^{3s+1} Q^3 &= 2^{3n} 3^{3m} P^3. \end{aligned}$$

Then by the uniqueness of the exponents of the prime number 3 we must have

$$\begin{aligned} 3s + 1 &= 3m \\ 1 &= 3m - 3s \\ 1 &= 3(m - s) \end{aligned}$$

this implies that 1 is divisible by 3; but that contradicts the theorem that says that if a and b are positive and $a|b$, then $a \leq b$. Since our assumption gives us a contradiction it follows that the assumption is false and the theorem is true. □

[Note that the argument is valid in the case that one of n, m, r, s is zero.]

Theorem. For each positive integer n:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Proof.

$$\begin{aligned} \sum_{i=1}^{1} \frac{1}{i(i+1)} &= \frac{1}{1(1+1)} = \frac{1}{2}; \\ \frac{1}{1+1} &= \frac{1}{2}. \end{aligned}$$

Therefore the statement is true for $n = 1$. We proceed by induction. The induction hypothesis is that for each positive integer n we have

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Therefore:

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} \\ &= \frac{n^2+2n+1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2}. \end{aligned}$$

Where the second step follows from the induction hypothesis. And the remaining steps complete the proof by induction. ∎

Theorem. If x is an integer then $x^2 \geq 0$.

Proof. By axiom D1 there are three cases: (1) $x = 0$, (2) $0 < x$, (3) $x < 0$:

Case 1. $x = 0$:

$$\begin{array}{rcll} 0 \cdot 0 & = & 0 & \text{by Theorem} \\ 0 \cdot 0 & \geq & 0 & \text{Assumption of case (1)} \end{array}$$

Case 2. $0 < x$:

$$\begin{array}{rcll} 0 & < & x & \text{Assumption of case (2)} \\ 0 \cdot x & < & x \cdot x & \text{Axiom D5} \\ 0 & < & x^2 & \text{Theorem, definition of notation} \\ x^2 & > & 0 & \text{notation} \\ \therefore x^2 & \geq & 0 & \end{array}$$

Case 3. $x < 0$:

$$\begin{array}{rcll} x & < & 0 & \text{Assumption of case (3)} \\ x + -x & < & 0 + -x & \text{Axiom D4} \\ 0 & < & -x & \text{additive inverse, additive identity} \\ (-x) \cdot (-x) & > & 0 & \text{case 2 above} \\ - - x \cdot x & = & - - x^2 & \text{Theorem and notation} \\ - - x^2 & = & x^2 & \text{Theorem} \\ x^2 & > & 0 & \text{from steps 4, 5 and 6 of case 3} \\ \therefore x^2 & \geq & 0. & \end{array}$$

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