Math 3100 Spring 2025 Test 01

February 21, 2025

Show all your work for each problem, if the work is incomplete or incorrect you may not receive full credit for that problem. If you do scratch work, indicate what is scratch work; no credit will be taken off for errors in the scratch work; partial credit will be given if the reasoning is correct and only computational errors are made. **Do 5 of the following 6 problems**; you may do all 6 for extra credit.

Problem 1. Determine if the following pairs of logical statements are equivalent. Show why they are or are not equivalent.

a.) $P \vee (\sim Q)$ and $\sim((\sim P) \wedge Q)$.

b.) $P \Rightarrow (\sim Q)$ and $\sim(P \wedge Q)$.

Solution. Part (a)

P	Q	$\sim Q$	$P \vee (\sim Q)$	$\sim P$	$(\sim P) \wedge Q)$	$\sim((\sim P) \wedge Q)$
T	T	F	T	F	F	T
T	F	T	T	F	F	T
F	T	F	F	T	T	F
F	F	T	T	T	F	T

Since columns 4 and 7 match, the statements are equivalent.

Alternate solution: By de Morgan's theorem:

$$\begin{aligned}\sim((\sim P) \wedge Q) &= (\sim(\sim P)) \vee (\sim Q) \\ &= P \vee (\sim Q)\end{aligned}$$

Part (b)

P	Q	$\sim Q$	$P \Rightarrow (\sim Q)$	$P \wedge Q$	$\sim(P \wedge Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T

Since columns 4 and 6 match, the statements are equivalent.

Alternate solution: Recall that $P \to Q$ is equivalent to $(\sim P) \vee Q$; so $P \Rightarrow (\sim Q)$ is equivalent to $(\sim P) \vee (\sim Q)$ so

$$(\sim P) \vee (\sim Q) \quad = \quad \sim (P \wedge Q))$$

and so the two are equivalent.

□

Problem 2. Prove the following theorem about the integers $\mathbb{Z}$ using only the axioms of the integers. Make sure to give a reason for each step.

Theorem: The additive inverse of an integer is unique. [Equivalently: if x is an additive inverse of n then $x = -n$.]

Proof.

$$\begin{aligned} x + n = & \quad 0 && \text{assumpton that } x \text{ is an inverse of } n \\ x + n + -n = & \quad 0 + -n && \text{closure, existence of additive inverse} \\ x + 0 = & \quad 0 + -n && \text{property of of additive inverse} \\ x = & \quad -n && \text{property of of additive identity .} \end{aligned}$$

Therefore, if x is an additive inverse of n, then it must equal $-n$.

□

Problem 3. Prove that for each positive integer n (where F_n is the n^{th} Fibonacci number):

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

Proof. We prove the statement by induction.

$$\begin{aligned} \sum_{i=1}^{1} F_i &= 1 \text{ and} \\ F_{1+2} - 1 &= F_3 - 1 = 2 - 1 = 1. \end{aligned}$$

Therefore, the formula works in the basic instance. Our induction hypothesis is

$$S_n : \sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

Now we consider the $n + 1$ case:

$$\begin{aligned} \sum_{i=1}^{n+1} F_i &= \sum_{i=1}^{n} F_i + F_{n+1} \\ &= F_{n+2} - 1 + F_{n+1} \\ &= F_{n+2} + F_{n+1} - 1 \\ &= F_{n+3} - 1 \\ &= F_{(n+1)+2} - 1. \end{aligned}$$

Which means that $S_n \Rightarrow S_{n+1}$. So the formula is true for all positive integers by the induction axiom. □

Problem 4. Prove that $\sqrt[5]{27}$ is irrational.

Proof. We assume that the statement is false and that there are integers a and b so that

$$\sqrt[5]{27} = \frac{a}{b}.$$

From the fundamental theorem of arithmetic, se can assume that

$$\begin{aligned} a &= 3^k Q \\ b &= 3^n P \end{aligned}$$

where Q and P are the product of primes none of which is 3 and where k or n are non-negative integers and can possibly be 0. Then we have

$$\begin{aligned} \sqrt[5]{27} &= \frac{a}{b} \\ b\sqrt[5]{27} &= a \\ b^5 3^3 &= a^5 \\ 3^{5n} P^5 3^3 &= 3^{5k} Q^5 \\ 3^{5n+3} P^5 &= 3^{5k} Q^5. \end{aligned}$$

The fundamental theorem tells us that the exponents of the prime number 3 on the two sides of the equation must be equal:

$$\begin{aligned} 5n+3 &= 5k \\ 3 &= 5k-5n. \end{aligned}$$

Since 5 divides the right side, this implies that $5|3$; but this contradicts the theorem that says that if $m|\ell$ then $m \leq \ell$. □

Problem 5. Consider the relation $\sim$ on the integers Z defined as follows:
$x \sim y$ means that $7|2x-2y$.

(a)Argue that this relation is equivalent to the relation $x \sim_* y$ where this means $7|x-y$.

(b)Prove that $\sim$ (or $\sim_*$) is an equivalence relation.

(c) What are the elements of $[3]_\sim$.

Solution. Part (a): If $7|2(x-y)$, then since 7 is a prime number and $7 \nmid 2$, then $7|(x-y)$. For the implication in the other direction, if $7|(x-y)$ then clearly $7|2(x-y)$. so the statements are equivalent.

Part (b).
Reflexive: $7|0$, so $7|(2x-2x)$. Then $x \sim x$.

Symmetric: If $x \sim y$, then $7|2(x-y)$, so $2(x-y)=7q$ for some $q \in \mathbb{Z}$. Then $2(y-x)=7(-q)$, So $7|(y-x)$ and $y \sim x$.

Transitive: If $x \sim y$, then $7|2(x-y)$, so $2(x-y)=7q$ for some $q \in \mathbb{Z}$, and if $y \sim z$, then $7|2(y-z)$, so $2(y-z)=7p$ for some $p \in \mathbb{Z}$, then

$$\begin{aligned} 2(x-y) &= 7q \\ 2x-2y &= 7q \\ 2y &= 2x-7q. \end{aligned}$$

Similarly

$$\begin{aligned} 2(y-z) &= 7p \\ 2y-2z &= 7p \\ (2x-7q)-2z &= 7p \\ 2x-2z &= 7p+7q = 7(p+q). \end{aligned}$$

So $7|2(x-z)$ and therefore $x \sim z$.

Part (c).

$$[3] = \{\ldots -18, -11, -4, 3, 10, 17, \ldots\}.$$

□

Problem 6. Suppose that each of a and b is a positive integer and S is the set $\{ax+by | ax+by > 0 \text{ for } x, y \in \mathbb{Z}\}$.

(a) Prove that S is non-empty.

(b) Prove that if d is the least element of S, then d divides a.

Proof. Part (a). If $x = 1$ and $y = 0$, then $ax + by = a$ and $a > 0$ by assumption. So $S \neq \emptyset$.

Part (b). Suppose that d does not divide a. Then there exist integers x and y so that $d = ax + by$. From the division algorithm, there are integers q and r so that $a = dq + r$ where $0 \leq r < d$. If $r = 0$ then $a = dq$ and d divides a, which contradicts our assumption. So we can assume that $0 < r < d$. Then substituting for d we have,

$$\begin{aligned} a &= dq + r \\ a &= (ax+by)q + r \\ a-(ax+by)q &= r \\ a(1-xq)+b(-yq) &= r. \end{aligned}$$

So $r \in S$. But since $0 < r < d$ we have that d was not the least member of the set. so our assumption that d does not divide a leads to a contradiction, so d divides a.

□