Logic: Propositional Calculus

Undefined terms: Statements and statement variables; a set of logical values $\{T, F\}$; operators $\vee, \wedge, \sim$.

Let $\mathbb{S}$ denote the collection of statements and assume in the following that each of P, Q and R is a statement.

Axiom L0. To any statement P a logical value can be assigned. Two sentences are logically equivalent if they have the same truth values for the same truth values of their clauses.

Definition. The statement P is said to be true if and only if it has truth value T; the statement P is said to be false if and only if it has truth value F.

Interpretations: The statement $P \vee Q$ is interpreted to mean "P or Q"; the statement $P \wedge Q$ is interpreted to mean "P and Q"; and the statement $\sim P$ is interpreted to mean "not P". These interpretations should be consistent with your understanding of the grammar of our (English in our case) language.

Axiom L1.

P	$\sim P$
T	F
F	T

Axiom L2.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Axiom L3.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Theorem 1.1. If P is a statement then:

$$\begin{array}{ll} \text{i.} & P \vee P = P \\ \text{ii.} & P \wedge P = P \\ \text{iii.} & \sim(\sim P) = P \end{array}$$

Theorem 1.2. The operators $\vee$ and $\wedge$ are commutative: If each of P and Q is a statement then:

$$\begin{array}{ll} \text{i.} & P \vee Q = Q \vee P \\ \text{ii.} & P \wedge Q = Q \wedge P \end{array}$$

Theorem 1.3. The operators $\vee$ and $\wedge$ are associative: If each of P, Q and R is a statement then:

$$\begin{array}{ll} \text{i.} & P \vee (Q \vee R) = (P \vee Q) \vee R \\ \text{ii.} & P \wedge (Q \wedge R) = (P \wedge Q) \wedge R \end{array}$$

Theorem 1.4. Each of the operators $\vee$ and $\wedge$ distributes over the other: If each of P, Q and R is a statement then:

$$\begin{array}{ll} \text{i.} & P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R) \\ \text{ii.} & P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \end{array}$$

Theorem 1.5. [De Morgan's Law for logic.] If each of P and Q is a statement then:

$$\begin{array}{ll} \text{i.} & \sim(P \vee Q) = (\sim P) \wedge (\sim Q) \\ \text{ii.} & \sim(P \wedge Q) = (\sim P) \vee (\sim Q) \end{array}$$

Definition. If each of P and Q is a statement then the statement $P \Rightarrow Q$ [read "P implies Q"] has the following truth values.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Theorem 1.6. If each of P and Q is a statement then the statement $P \Rightarrow Q$ is equivalent to the statement $(\sim P) \vee Q$.