Homomorphisms.

Definition. If $(G_1, *)$ and $(G_2, \diamond)$ are sets with operations $*$ and $\diamond$ respectively and $F : G_1 \to G_2$ is a function, then F is called a *homomorphism* if and only if

$$F(x * y) \quad = \quad F(x) \diamond F(y).$$

Definition. A homomorphism that is one-to-one is called an *isomorphism*.

Exercise 7.3. Prove that the following functions are homomorphisms (Caution: I think one of them is not!); which, if any, are isomorphisms:

a. $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ where $f(x) = 5x$.

b. $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ where $f(x) = 5x$.

c. $f : (\mathbb{R}, \cdot) \to (\mathbb{R}, \cdot)$ where $f(x) = 5x$.

d. $f : (\mathbb{R}, +) \to (\mathbb{R}, \cdot)$ where $f(x) = 2^x$.

e. $f : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$ where $f(x) = \ln(x)$.

f. $f : (\mathbb{C}, +) \to (\mathbb{R}, +)$ where $f(x + yi) = 3x + 5y$ ($\mathbb{C}$ denotes the complex numbers).

Exercise 7.4. Determine if the following are homomorphisms (isomorphisms), in each case make sure to check to see if the function is well-defined:

a. $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_5, +_5)$ where $F([x]_5) = [2x + 1]_5$.

b. $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_5, +_5)$ where $F([x]_5) = [2x]_5$.

c. $F : (\mathbb{Z}_{10}, +_{10}) \to (\mathbb{Z}_5, +_5)$ where $F([x]_5) = [2x]_5$.

d. $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_{10}, +_{10})$ where $F([x]_5) = [2x]_5$.

e. $F : (\mathbb{Z}_{31}, +_{31}) \to (\mathbb{Z}_{31}, +_{31})$ where $F([x]_{31}) = [7x]_{31}$.

f. $F : (\mathbb{Z}_5, \cdot_5) \to (\mathbb{Z}_5, \cdot_5)$ where $F([x]_5) = [2x + 1]_5$.

g. $F : (\mathbb{Z}_5, \cdot_5) \to (\mathbb{Z}_5, \cdot_5)$ where $F([x]_5) = [2x]_5$.

h. $F : (\mathbb{Z}_4, +_4) \to (\mathbb{Z}_5, \cdot_5)$ where $F([x]_4) = [2]_5^x$.

Exercise 7.5. Suppose $F : (\mathbb{Z}_5, +_5) \to (\mathbb{Z}_n, +_n)$ where $F([x]_5) = [ax + b]_n$ is a homomorphism. Then:

1.) Either a or n is divisible by 5.

2.) $b \sim_n 0$.

3.) $F([0]_5) = [0]_n$.

Theorem 7.7 Suppose $F : (\mathbb{Z}_m, +_m) \to (\mathbb{Z}_m, +_m)$ is a homomorphism. Then $F([0]_m) = [0]_n$.

Exercise 7.6. Let S_n denote the permutation group on the elements $\{1, 2, 3, \ldots, n\}$. Determine if the following are homomorphisms: in each case make sure to check to see if the function is well-defined:

a. $F : (\mathbb{Z}_5, +_5) \to (S_5, \cdot)$ where $F([x]_5) = (12345)^x$,
b. $F : (\mathbb{Z}_6, +_6) \to (S_6, \cdot)$ where $F([x]_6) = (123456)^x$,
c. $F : (\mathbb{Z}_5, +_5) \to (S_6, \cdot)$ where $F([x]_5) = (123456)^x$,
d. $F : (\mathbb{Z}_6, +_6) \to (S_5, \cdot)$ where $F([x]_6) = ((12)(345))^x$,
e. $F : (\mathbb{Z}_5 - \{0\}, \cdot_5) \to (S_4, \cdot)$ where $F([x]_5) = (1234)^x$,
f. $F : (\mathbb{Z}_8, +_8) \to (S_4, \cdot)$ where $F([x]_8) = (1234)^x$.