Math 3100 Summer 2021
Project 01
Key

Part I Predict Logic.

Comments on notation: if $R(x, y)$ is a statement then

$$\exists x \forall y R(x, y)$$

is parsed as follows:

$$\exists x(\forall y(R(x, y))).$$

Another way of thinking about this is as follows: Let $H(x)$ be the statement $\forall y R(x, y)$ then $\exists x \forall y R(x, y)$ is equivalent to $\exists x H(x)$. We could write this as

$$\exists x \forall y R(x, y) \quad = \quad \exists x H(x),$$

or

$$\exists x \forall y R(x, y) \quad \Leftrightarrow \quad \exists x H(x).$$

For example if $R(x, y)$ is the statement $x < 1 + y^2$, then $R(x, y)$ is true because if we let $x = 0.5$ then $0.5 < 1 + y^2$ no matter what y is; so $H(0.5)$ is always true this says that $\forall y R(0.5, y)$. And since the statement is true for $x = 0.5$ it follows that there is a value of x for which the statement is true, therefore

$$\exists x \forall y R(x, y).$$

We can also conclude that if a statement is true for all y, then there obviously exists a y value (in fact any will do) that makes the statement true. Thus

$$\exists x \forall y R(x, y) \quad \Rightarrow \quad \exists x \exists y R(x, y).$$

Exercise 1.1. Argue that the following are true implications no matter what the statement $R(x, y)$ is. In each case give an example of a statement for $R(x, y)$ for which the right side is true but the left side is not - this proves that the implication arrows can't be proven to go the other way. (Hint: you might want to do this after Exercise 2.1 and 2.2 since these will produce answers for some of these.):

$$\begin{aligned}\forall x\forall yR(x,y) &\Rightarrow \exists x\forall yR(x,y);\\ \exists x\forall yR(x,y) &\Rightarrow \forall y\exists xR(x,y);\\ \forall y\exists xR(x,y) &\Rightarrow \exists x\exists yR(x,y).\end{aligned}$$

Note we may sometimes rewrite this as:

$$\forall x\forall yR(x,y) \Rightarrow \exists x\forall yR(x,y) \Rightarrow \forall y\exists xR(x,y) \Rightarrow \exists x\exists yR(x,y).$$

Solution. The statement $\forall x\forall yR(x,y)$ is true whenever $R(x,y)$ is true no matter what values x and y are. So, if $R(x,y)$ is true for all x and y, then it's true for one specific x and all y, in fact, any x will do. Therefore,

$$\forall x\forall yR(x,y) \Rightarrow \exists x\forall yR(x,y).$$

The statement $\exists x\forall yR(x,y)$ says that there is an x value, say $x = x_0$ so that $R(x_0, y)$ is true no matter what values of y are chosen. The statement $\forall y\exists xR(x,y)$ says that for each y value, there is an x value (which would usually depend on y), say $x = x_y$ so that $R(x_y, y)$ is true no matter what the value of y. Therefore,

$$\exists x\forall yR(x,y) \Rightarrow \forall y\exists xR(x,y).$$

The statement $\forall y\exists xR(x,y)$ says that for each y value, there is an x value say $x = x_y$ so that $R(x_y, y)$ is true no matter what the values y are. Therefore, since the statement $R(x_y, y)$ is true for all y, it's true for a specific y, say y_0. Therefore the statement $R(x_{y_0}, y_0)$ is true, so $\exists x\exists yR(x,y)$. Therefore

$$\forall y\exists xR(x,y) \Rightarrow \exists x\exists yR(x,y).$$

□

Exercise 1.2. Similarly, argue that the following are true implications no matter what the statement $R(x,y)$ is and that in each case the implication does not go in the other direction.

$$\begin{aligned}\forall y\forall xR(x,y) &\Rightarrow \exists y\forall xR(x,y);\\ \exists y\forall xR(x,y) &\Rightarrow \forall x\exists yR(x,y);\\ \forall x\exists yR(x,y) &\Rightarrow \exists y\exists xR(x,y).\end{aligned}$$

Solution. Just interchange the role of x and y in the previous solution (for 1.1) and you obtain a solution for exercise 1.2. □

For exercises 2.1 and 2.2 you may use all the facts that you know about the real numbers.

Exercise 2.1. With the understanding that $\forall y \forall x R(x,y) = \forall x \forall y R(x,y)$, and similarly for the $\exists$ symbol, there are six permutations of the allowable statements with symbols $\forall$ and $\exists$ and variables x and y. For the following statement $R(x,y) : 3x - 4y = 10$, determine which of the six are true and which are false and argue in each case why.

Solution. For $R(x,y) : 3x - 4y = 10$ we have

$$\forall y \forall x R(x,y) \text{ is false;} \tag{1}$$
$$\exists x \forall y R(x,y) \text{ is false;} \tag{2}$$
$$\forall y \exists x R(x,y) \text{ is true;} \tag{3}$$
$$\exists y \exists x R(x,y) \text{ is true;} \tag{4}$$
$$\exists y \forall x R(x,y) \text{ is false;} \tag{5}$$
$$\forall x \exists y R(x,y) \text{ is true.} \tag{6}$$

1. Since $R(1,1)$ is false, then statement (1) is false.

2. Since there is no single value of x so that for every y we have $3x-4y = 10$, then (2) is false.

3. Since $x = (10+4y)/3$ makes the statement $3x - 4y = 10$ true it follows that $R((10+4y)/3, y)$ is true so (3) is true. Notice in this case, the existence of the x depends on which value of y is chosen for the $\forall y$ clause of the statement.

4. Observe that the statement $R(\frac{10}{3}, 0)$ is true so (4) is true.

5. Since there is no single value of y so that for every x we have $3x-4y = 10$, then (5) is false.

6. Observe that $R(x, (3x-10)/4)$ is true, so (6) is true. □

Exercise 2.2. Repeat exercise 2.2 for the statement $R(x,y) : x^2 + y > 1$.

Solution. For $R(x,y) : x^2 + y > 1$ we have

$$\begin{aligned} \forall y \forall x R(x,y) &\quad \text{is} \quad \text{false}; & (7) \\ \exists x \forall y R(x,y) &\quad \text{is} \quad \text{false}; & (8) \\ \forall y \exists x R(x,y) &\quad \text{is} \quad \text{true}; & (9) \\ \exists y \exists x R(x,y) &\quad \text{is} \quad \text{true}; & (10) \\ \exists y \forall x R(x,y) &\quad \text{is} \quad \text{true}; & (11) \\ \forall x \exists y R(x,y) &\quad \text{is} \quad \text{true}. & (12) \end{aligned}$$

7. Since $R(\frac{1}{2}, \frac{1}{2})$ is false, then statement (7) is false.

8. There is no single value of x so that for every y we have $3x^2 + y > 1$; because if there were such an x, setting $y = -2 - x^2$ makes the statement $R(x,y)$ false. So (8) is false.

9. Since $x = |y| + 2$ makes the statement $3x^2 + y > 1$ true it follows that $R(|y| + 2, y)$ is true so (9) is true.

10. Observe that the statement $R(1,1)$ is true so (10) is true.

11. Since for $y = 2$ the statement $x^2 + 2 > 1$ is true no matter what x value is chosen, (11) is true.

12. Observe that $R(1,1)$ is true, so (12) is true. □

Finally I asked so show that the following hold:

$$\begin{aligned} \exists y \forall x R(x,y) &\not\Rightarrow \forall y \forall x R(x,y); & (13) \\ \forall x \exists y R(x,y) &\not\Rightarrow \exists y \forall x R(x,y); & (14) \\ \exists y \forall x R(x,y) &\not\Rightarrow \forall x \exists y R(x,y). & (15) \end{aligned}$$

Let $R_1(x,y) : 3x - 4y = 10$ and $R_2(x,y) : x^2 + 2 > 1$. Then:

13. Using R_2, we see that (11) is true and (7) is false, so this is a valid counterexample and so verifies (13).

14. To show the counterexample that verifies (14) we use R_2 but with the roles of x and y interchanged. (I.e.: we use $R_2(y,x) : y^2 + x > 1$.) We see that (11) is true but (8) is false. So interchanging the roles of x and y we have verified (14).

15. Using $R_1(x,y)$ we see that (5) is true but (4) is false, so this example verifies (15).

Part II Theorems about the integers.

In this section you are asked to prove some theorems about the integers. You may only use the axioms of the integers to prove these statements (and none of the theorems from the 02 notes). You may use a theorem that you proved to prove a later theorem. You may also make up and prove any lemmas that you think would be useful in your proofs.

Exercise 3. Prove the following theorems about the integers from the axioms.

a.) If each of a, b, c and d is an integer, then

$$((a+b)+c)+d \;=\; (a+c)+(b+d).$$

Proof.

$$\begin{aligned} ((a+b)+c)+d &= ((a+(b+c))+d && \text{Assoc. Add.} \\ &= ((a+(c+b))+d && \text{Comm. Add.} \\ &= ((a+c)+b)+d && \text{Assoc. Add.} \\ &= (a+c)+(b+d) && \text{Assoc. Add.} \end{aligned}$$

□

b.) If each of a, b, x and y is an integer, then

$$(a(x+y))b \;=\; a(bx)+a(by).$$

Proof.

$$\begin{aligned} (a(x+y))b &= (ax+ay)b && \text{Dist.} \\ &= b(ax+ay) && \text{Comm. Mult.} \\ &= b(ax)+b(ay) && \text{Dist.} \\ &= (ba)x+(ba)y && \text{Assoc. Mult. (twice)} \\ &= (ab)x+(ab)y && \text{Comm. Mult. (twice)} \\ &= a(bx)+a(by) && \text{Assoc. Mult. (twice).} \end{aligned}$$

□

From this point on you may assume that the quantities $a+b+c$ and abc are well defined (as explained in class) and that $ab+c$ means $(a \cdot b)+c$.

c.) If x is an integer, then

$$x + x \quad = \quad 2x$$

where the symbol 2 is defined to be the integer equal to $1+1$.

Proof.

$$\begin{aligned} x + x &= x1 + x1 && \text{Mult. Id. (twice)} \\ &= x(1+1) && \text{Dist.} \\ &= x(2) && \text{definition} \\ &= 2x && \text{Comm. Mult.} \end{aligned}$$

□

d.) If x is an integer, then

$$(x + (-1)) \cdot (x + 1) \quad = \quad x^2 - 1.$$

Where $x^2 - 1$ means $x^2 + (-1)$ as in the notes and x^2 means $x \cdot x$.

Proof.

$$\begin{aligned} (x+(-1)) \cdot (x+1) &= (x+(-1))x + (x+(-1))1 && \text{Dist.} \\ &= (x+(-1))x + (x+(-1)) && \text{Mult. Id.} \\ &= x(x+(-1)) + (x+(-1)) && \text{Comm. Mult.} \\ &= (xx + x(-1)) + (x+(-1)) && \text{Dist.} \\ &= (x^2 + x(-1)) + (x+(-1)) && \text{notation} \\ &= ((x^2 + x(-1)) + x) + (-1) && \text{Assoc. Add.} \\ &= (x^2 + (x(-1) + x)) + (-1) && \text{Assoc. Add.} \\ &= (x^2 + ((-1)x + x)) + (-1) && \text{Comm. Mult.} \\ &= (x^2 + (-x + x)) + (-1) && \text{Lemma/Theorem.} \\ &= (x^2 + 0)) + (-1) && \text{Add. Inverse.} \\ &= x^2 + (-1) && \text{Add. Id.} \\ &= x^2 - 1 && \text{notation.} \end{aligned}$$

The fact that $(-1)x = -x$ needs to be proven, via a lemma or a theorem. I overlooked this when I made out the problem, it makes the problem harder than I meant. I will not take off credit if this was overlooked and I'll give extra credit to anyone who stated and proved the lemma. □

e.) If each of a and b is an integer then there exists a unique integer x so that

$$a + x = b.$$

Proof. Two things need to be proven: existence and uniqueness. First I'll prove existence.

Let $x = -a + b$, this follows from the existence of the additive inverse and closure. Then:

$$\begin{aligned} a + x &= a + (-a + b) && \text{substitution} \\ &= (a + -a) + b && \text{Assoc. Add.} \\ &= (-a + a) + b && \text{Comm. Add.} \\ &= 0 + b && \text{Add. Inverse} \\ &= b + 0 && \text{Comm. Add.} \\ &= b && \text{Add. Id.} \end{aligned}$$

This proves existence. To prove uniqueness, suppose that $\hat{x}$ is such that $a + \hat{x} = b$. Then,

$$\begin{aligned} a + \hat{x} &= b && \text{assumption} \\ -a + (a + \hat{x}) &= -a + b && \text{Closure} \\ (-a + a) + \hat{x} &= -a + b && \text{Assoc. Add.} \\ 0 + \hat{x} &= -a + b && \text{Add. Inverse} \\ \hat{x} + 0 &= -a + b && \text{Comm. Add.} \\ \hat{x} &= -a + b && \text{Add. Id.} \\ \hat{x} &= x && \text{definition.} \end{aligned}$$

Therefore x is unique.

□