Math 3100 Test 01.

The test is due by midnight Monday June 21. The test is open notes this includes my notes on the website. You may not receive any other outside assistance and may not discuss the quiz with anyone. Please affirm at the beginning of your hand-in work that you have abided by these conditions.

Email to me as an attachment your solutions to the problems as a pdf file with the file name beginning with your last name: e.g. smithxyztest01.pdf.

Problem 1.

Determine if the following statement is a tautology:

$$\sim (P \vee (\sim Q)) \Rightarrow (\sim P) \wedge Q.$$

Solution. Yes, it is a tautology. It's actually one of de Morgan's laws with Q replaced with $\sim Q$.

□

Problem 2.

Consider the following relation about the real numbers:

$$R(x, y) : \qquad 3x + 2y + 1 = 0.$$

Determine which of the following statements are true and argue why the false ones are not true.

a.) $\forall x \exists y R(x, y)$
b.) $\exists x \forall y R(x, y)$
c.) $\sim (\forall x \exists y R(x, y))$
d.) $\sim (\exists x \forall y R(x, y))$
e.) $\forall x \exists y (\sim R(x, y))$
f.) $\exists x \forall y (\sim R(x, y))$.

Solution.
(a) is true.

For each x the y that works to make it true is

$$y = \frac{-1 - 3x}{2}.$$

(b) is false.

There is no fixed x that makes $R(x,y)$ true for all y. For two different y's the x value that makes $R(x,y)$ true have to be different.

(c) is false.
Statement (c) is the negation of (a); since (a) is true (c) must be false.

(d) is true.
Statement (d) is the negation of (b) since (b) is false; (d) must be true.

(e) is true.
Statement (e) is the negation of (b) and so is equivalent to (d); (e) must be true.

(f) is false.
Statement (f) is the negation of (a) and so is equivalent to (c); (f) must be false.

Any correct reasoning on the ones that are false will be accepted. □

Problem 3.
Prove that the additive inverse of an integer is unique.

Solution. [There are many possible different proofs, I give one below.]

Let x be an integer and suppose that $\hat{x}$ is an additive inverse of x.

statement			reason	
$x + \hat{x}$	$=$	0	hypothesis/assumption	*
$x + (-x)$	$=$	0	additive inverse	
$x + (-x)$	$=$	$x + \hat{x}$	transitive property of =	
$(x + (-x)) + (-x)$	$=$	$(x + \hat{x}) + (-x)$	closure (& logic)	*
$0 + (-x)$	$=$	$(x + \hat{x}) + (-x)$	add. inverse	*
$-x$	$=$	$(x + \hat{x}) + (-x)$	add. identity	*
$-x$	$=$	$x + (\hat{x} + (-x))$	associativity of add.	
$-x$	$=$	$x + ((-x) + \hat{x})$	commutativity of add.	
$-x$	$=$	$(x + (-x)) + \hat{x}$	associativity of add.	
$-x$	$=$	$0 + \hat{x}$	add. inverse	*
$-x$	$=$	$\hat{x}$	add. identity	*

Therefore, since any additive inverse of x must be $-x$ and so is unique. [Note: The steps with * are the important steps, if the others were left off there was no loss of credit.] $\square$

Problem 4.

Show that for each $n \in \mathbb{N}$:

$$\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}.$$

Proof. We prove the theorem by induction. Let P_n be the statement

$$P_n: \qquad \sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}.$$

Then for P_1 we have:

$$\text{LHS}_1: \qquad \sum_{i=1}^{1} i(i+2) = 1 \cdot 3 = 3$$

$$\text{RHS}_1: \qquad \frac{1(1+1)(2+7)}{6} = \frac{18}{6} = 3.$$

So P_1 is true. Next we show that $P_n \Rightarrow P_{n+1}$. So P_n is our induction hypothesis. Then:

$$\begin{aligned}
\sum_{i=1}^{n+1} i(i+2) &= \sum_{i=1}^{n} i(i+2) + (n+1)(n+3) \\
&= \frac{n(n+1)(2n+7)}{6} + (n+1)(n+3) \\
&= \frac{n(n+1)(2n+7)}{6} + \frac{(n+1)(n+3)}{6} \\
&= \frac{n(n+1)(2n+7)}{6} + \frac{(n+1)(n+3)}{6} \\
&= \frac{(n+1)[n(2n+7)+6(n+3)]}{6} \\
&= \frac{(n+1)[2n^2+7n+6n+18]}{6} \\
&= \frac{(n+1)[2n^2+13n+18]}{6} \\
&= \frac{(n+1)[(n+2)(2n+9)]}{6} \\
&= \frac{(n+1)[((n+1)+1)(2(n+1)+7)]}{6}.
\end{aligned}$$

The second step follows from the induction hypothesis; the last step yields P_{n+1}. So the statement is true for all positive integers by the induction axiom. □

Problem 5.

Show that for each $n \in \mathbb{N}$:

$$4 \,\Big|\, (5^{4n} + 3).$$

Proof. We prove the theorem by induction. Let P_n be the statement

$$P_n : \quad 4 \,\Big|\, (5^{4n} + 3).$$

Then to consider P_1 we have:

$$5^{4\cdot 1} + 3 \;=\; 625 + 3 = 4(157).$$

This verifies P_1. Next we show that $P_n \Rightarrow P_{n+1}$. So P_n is our induction hypothesis. Then we assume that there is an integer q so that $5^{4n} + 3 = 4q$; and some simple algebra gives us $5^{4n} = 4q - 3$. So we have

$$\begin{aligned} 5^{4(n+1)} + 3 &= 5^{4n+4} + 3 \\ &= 5^{4n} \cdot 5^4 + 3 \\ &= (4q-3) \cdot 5^4 + 3 \\ &= 4q \cdot 5^4 - 3 \cdot 5^4 + 3 \\ &= 4q \cdot 5^4 - 1872 \\ &= 4(q \cdot 5^4 - 468). \end{aligned}$$

The third step follows from the induction hypothesis; the last step yields P_{n+1}. So the statement is true for all positive integers by the induction axiom. □

Problem 6.

Use the fundamental theorem of arithmetic (or the corollary in the notes) to prove that $\sqrt[3]{16}$ is irrational. Make sure to indicate your reasoning behind each step.

Solution. [There are many possible different proofs, I give one below.]

We assume that $\sqrt[3]{16}$ is rational and that a and b are positive integers so that

$$\sqrt[3]{16} = \frac{a}{b}.$$

Then the fundamental theorem tells us that there are unique non-negative integers n and k so that

$$\begin{aligned} a &= 2^n P \\ b &= 2^k Q \end{aligned}$$

where P and Q are integers that are not divisible by 2. Then we have

$$\begin{aligned} \sqrt[3]{16} &= \frac{a}{b} \\ \sqrt[3]{16} \cdot b &= a \\ 16b^3 &= a^3 \\ 2^4(2^kQ)^3 &= (2^nP)^3 \\ 2^4(2^{3k})Q^3 &= 2^{3n}P^3 \\ 2^{3k+4}Q^3 &= 2^{3n}P^3. \end{aligned}$$

Since 3 is prime and neither Q nor P are divisible by 3 then neither are Q^3 nor P^3. So by the uniqueness property.

$$\begin{aligned} 3k + 4 &= 3n \\ 4 &= 3n - 3k. \end{aligned}$$

But the right side is divisible by 3 and the left side is not. So this is a contradiction, therefore the assumption that $\sqrt[3]{16}$ is rational is false and hence $\sqrt[3]{16}$ is irrational. □

Problem 6.

For each step in the following outline of the proof of the theorem stated, show why the step is true. You may use the axioms of the integers and the theorems from the 02 notes.

Theorem. Suppose that each of p, a and b is a positive integer and there exists integers x and y (possibly negative) so that

$$1 = px + ay.$$

Then, if you have $p|ab$ it follows that $p|b$.

Proof. We assume that x and y are integers so that

$$1 = px + ay.$$

Step 1. There is a number q so that

$$ab = pq.$$

Step 1 follows from the hypothesis of the theorem when is says that $p|ab$.

Step 2 follows by multiplying both sides of step 1 by y.

$$yab = ypq.$$

To obtain step 3 we start with the given condition and multiply both sides by b:

$$\begin{aligned} 1 &= px + ay && \text{given} \\ b &= bpx + bay && \text{multiply by } b. \end{aligned}$$

then combining this with step 2 we get

$$\begin{aligned} b &= bpx + ypq \\ b &= p(bx + yq). \end{aligned}$$

From which we can conclude: Step 3 (conclusion).

$$p|b.$$

□