

Math 3100 Project 01.

The project is due Before Class July 30. The project is open notes, this includes my notes on the website. You may not receive any other outside assistance and may not discuss it with anyone.

Constructing the rationals from the integers.

Since we will be using the properties of the integers for our construction, for this work you may only use the axioms of the integers. In writing up the proofs for the theorems and exercises for the project, it will not be necessary for you to give reasons when you use the associative, commutative or distributive axioms; but otherwise make sure to state your reasons for your steps.

For the project: Prove theorems 7.2, 7.3, 7.5 and 7.6 stated below; also solve exercise 7.2; for theorem 7.4, just prove that \mathbb{Q} satisfies axioms B5 and C1.

[Hint: Although it is not necessary, it may be helpful in formulating proofs, for you to remember what I said in class about how the elements of \mathbb{Q} are related to the rational numbers as you learned about them in grammar school.]

As usual \mathbb{Z} is the set of integers. We will be working with the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$; this is the set of all integer pairs whose second element is non-zero. We define an equivalence relation \sim on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Definition. Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define \sim on S as follows:

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc.$$

Theorem 7.1. The relation \sim is an equivalence relation on S .

Proof Outline. We need to show that the relation \sim satisfies the conditions for being an equivalence relation.

1.) Reflexive: This means that $(a, b) \sim (a, b)$ and is easy.

2.) Symmetric: This one is easy too; write down what $(a, b) \sim (c, d)$ means and what $(c, d) \sim (a, b)$ means and compare.

3.) Transitive: This is the tricky one. Given $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ we want to conclude that $(a, b) \sim (e, f)$. Let's "translate" using the definition:

Given $ad = bc$ and $cf = de$ we want to conclude that $af = be$. Now if we were in the reals it would be easy; solve the second equation for d and substitute into the first and do the algebra. But this involves division and we don't have division for the integers. Recall that the cancellation rule can be sometimes used in place of division and use that to complete the proof. \square

Notation. The symbol \mathbb{Q} will be used to denote the collection of equivalence classes:

Exercise 7.1. Determine list four or five elements of each of the following equivalence class:

- a.) $[(0, 1)]$
- b.) $[(2, 1)]$
- c.) $[(1, 2)]$
- d.) $[(2, 3)]$
- e.) $[(3, 2)]$
- f.) $[(1, 1)]$
- g.) $[(-1, 3)]$
- h.) $[(1, -3)]$.

Definition. The operations $+$ and \cdot are defined as follows:

$$\begin{aligned} [(a, b)] + [(c, d)] &= [(ad + bc, bd)] \\ [(a, b)] \cdot [(c, d)] &= [(ac, bd)]. \end{aligned}$$

[Comment: The $+$ on the right operates on elements of S whereas the one on the left operates on integers; this shouldn't cause any confusion since

the plus signs will always be between the same type of elements. It's sort of like, when you were in 2660 you saw the instructor use the symbol \cdot to indicate multiplication of matrices even though they also used it to indicate multiplication of numbers. If I am concerned that there may be the possibility of confusion, I will denote the operations on \mathbb{Q} as $+\sim$ and $\cdot\sim$.]

Theorem 7.2. The operations $+\sim$ and $\cdot\sim$ are well-defined.

Exercise 7.2. The operation \oplus defined as follows is not well-defined.

$$[(a, b)] \oplus [(c, d)] = [(a + c, b + d)].$$

Theorem 7.3. There exist additive and multiplicative identities associated with the operations of addition $+$ and multiplication \cdot on \mathbb{Q} .

Theorem 7.4. The A, B and C axioms of the integers are satisfied by \mathbb{Q} . [Most of these will be straight forward, the existence of the multiplicative and additive identities is claimed by theorem 7.3; given $[(a, b)]$ you will have to determine its additive inverse. This is really 11 theorems in 1, one for each axiom.]

Theorem 7.5 A. If $[(a, b)] \in \mathbb{Q}$ then $[(a, b)]$ has an additive inverse.

Theorem 7.5 B. If $[(a, b)] \neq [(0, 1)]$ then $[(a, b)]$ has a multiplicative inverse.

Theorem 7.6. Let $\varphi : (\mathbb{Z}, +) \rightarrow (\mathbb{Q}, +\sim, \cdot\sim)$ be defined as follows:

$$\varphi(n) = [n, 1].$$

Then φ is a homomorphism with respect to both addition $+$ and $+\sim$ and multiplication \cdot and $\cdot\sim$ operators.