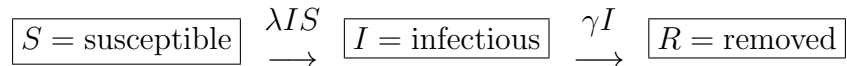


Kermack-McKendrick (SIR) Model of Infectious Diseases.

We suppose that we have a population S of susceptible hosts for a disease, a population I of infectious host and a population R of recovered hosts. Roughly the situation may be flow charted as in the figure below:



Simplified SIR Model 01:

For our first simplified model we will assume:

- No births (or equivalently, birth rates = death rates);
- neither immigration nor emigration;
- removed = 0;
- no loss of immunity.

This simplifies the flow chart to the following:



These assumptions imply $N(t)$ is constant, $N(t) = N(0) = N_0$. Each of the variables S, I, R is a function of time t where, in our cases, t will be in days.

For some quantity A let ΔA denote the change of quantity A from day m at midnight to day $m + 1$ at midnight; so ΔA denotes the “(average) change of quantity A per day”.

$$\begin{aligned} \Delta S &= -\text{new infections} \\ \Delta I &= \text{new infections} - \text{new recovered} \\ \Delta R &= \text{new cures.} \end{aligned}$$

We have:

$$\Delta I = -\Delta S - \Delta R.$$

To set up the differential equations we consider an encounter between a susceptible and an infectious. We assume that there is a constant probability α of infection with each encounter. So assume an average of n encounters per day between members of the population; a particular infectious individual will infect $n\alpha \frac{S}{N_0}$ individuals per day. (Note, there is the possibility of two infectious encountering the same susceptible, this is on the order of $(\frac{n}{N_0})^2$ and for large populations $n \ll N_0$ this

quantity, along with higher order multiple encounters is negligible.) So the total number of individuals infected by the infectious group per day is

$$n\alpha \frac{S}{N_0} I.$$

Let $\lambda = \frac{n\alpha}{N_0}$. This gives us equation (1):

$$(1) \quad \frac{dS}{dt} = S' = -\lambda IS.$$

Assume that infectious individuals become cured at a constant rate proportional to the number of infectious individuals; assume this happens with proportionality constant γ so that gives us equation (2)

$$(2) \quad \frac{dR}{dt} = R' = \gamma I.$$

Finally since the change of I is the change of S moving into I and minus the change of R coming from I we have equation (3) and it's equivalent, equation (4):

$$\begin{aligned} \frac{dI}{dt} = I' &= -\frac{dS}{dt} - \frac{dR}{dt} \\ &= -S' - R' \\ (3) \quad &= \lambda IS - \gamma I \\ (4) \quad &= (\lambda S - \gamma)I. \end{aligned}$$

Following are some conclusions that can be made regarding this system of equations without solving them. Observe that by assumption $N(t)$ is constant so:

$$\begin{aligned} N(t) &= N(0) = N_0 \\ S(t) + I(t) + R(t) &= N_0 \\ S'(t) + I'(t) + R'(t) &= 0. \end{aligned}$$

Since S, I and λ are all positive, equation (1) tells us that $S'(t) < 0$ so,

$$\begin{aligned} S'(t) &< 0 \\ \therefore S(t) &< S_0 \text{ for all } t > 0 \\ \text{So } \lambda S - \gamma &< \lambda S_0 - \gamma \\ (\lambda S - \gamma)I &< (\lambda S_0 - \gamma)I \\ I' &< (\lambda S_0 - \gamma)I. \end{aligned}$$

If $I' < 0$ then the epidemic fizzles out. So using this result with equation (3) from above we have that if $\lambda S_0 - \gamma < 0$ then the epidemic

fizzles; this happens when

$$\begin{aligned}\lambda S_0 - \gamma &< 0 \\ \lambda S_0 &< \gamma \\ S_0 &< \frac{\gamma}{\lambda}.\end{aligned}$$

For a fixed disease we have no control over the constant γ . But we have some control over the “constant” λ : since we assume that $N(t)$ is the constant N_0 we have

$$\lambda = \frac{n\alpha}{N_0}$$

where, again, α is a constant related to the disease. But we can decrease α by making it less likely that when a susceptible meets an infectious that there will be a new infection by using masks and social distancing. Assuming we do this to obtain α' a smaller α , then since we want $I' < 0$. So we want:

$$\begin{aligned}S_0 &< \frac{\gamma}{\lambda} \\ S_0 &< \frac{\gamma N_0}{n\alpha'} \\ n\alpha' S_0 &< \gamma N_0 \\ n &< \frac{\gamma N_0}{\alpha' S_0}.\end{aligned}$$

Assuming $S_0 \approx N_0$ so that $\frac{N_0}{S_0} \approx 1$ this gives us a way to combat an epidemic. Obviously the smaller the value of n , the smaller the average number of interactions between members of our population, the faster the epidemic will end. But we must at least have $n < \frac{\gamma}{\alpha'}$ in order to have any hope that the epidemic can be stopped. So the strategy of decreasing n by sequestering enough individuals well below the $n < \frac{\gamma}{\alpha'}$ threshold will eventually cause the epidemic to end.