**Kermack-McKendrick (SIR) Model of Infectious Diseases**.

We suppose that the have a population $S$ of susceptible hosts for a disease, a population $I$ of infectious host and a population $R$ of recovered hosts. Roughly the situation may be flow charted as in the figure below:

$$\boxed{S = \text{susceptible}} \overset{\lambda IS}{\longrightarrow} \boxed{I = \text{infectious}} \overset{\gamma I}{\longrightarrow} \boxed{R = \text{removed}}$$

**Simplified SIR Model 01:**

For our first simplified model we will assume:

- No births (or equivalently, birth rates = death rates);
- neither immigration nor emigration;
- removed = 0;
- no loss of immunity.

This simplifies the flow chart to the following:

$$\boxed{S = \text{susceptible}} \longrightarrow \boxed{I = \text{Infective}} \longrightarrow \boxed{R = \text{Recovered}}$$

These assumptions imply $N(t)$ is constant, $N(t) = N(0) = N_0$. Each of the variables $S, I, R$ is a function of time $t$ where, in our cases, $t$ will be in days.

For some quantity $A$ let $\Delta A$ denote the change of quantity $A$ from day $m$ at midnight to day $m+1$ at midnight; so $\Delta A$ denotes the "(average) change of quantity $A$ per day".

$$\begin{aligned}
\Delta S &= -\text{new infections} \\
\Delta I &= \text{new infections} - \text{new recovered} \\
\Delta R &= \text{new cures.}
\end{aligned}$$

We have:

$$\Delta I = -\Delta S - \Delta R.$$

To set up the differential equations we consider an encounter between a susceptible and an infectious. We assume that there is a constant probability $\alpha$ of infection with each encounter. So assume an average of $n$ encounters per day between members of the population; a particular infectious individual will infect $n\alpha\frac{S}{N_0}$ individuals per day. (Note, there is the possibility of two infectious encountering the same susceptible, this is on the order of $(\frac{n}{N_0})^2$ and for large populations $n << N_0$ this quantity, along with higher order multiple encounters is negligible.) So the total number of individuals infected by the infectious group per day is

$$n\alpha \frac{S}{N_0} I.$$

Let $\lambda = \frac{n\alpha}{N_0}$. This gives us equation (1):

$$\frac{dS}{dt} = S' \quad = \quad -\lambda I S. \tag{1}$$

Assume that infectious individuals become cured at a constant rate proportional to the number of infectious individuals; assume this happens with proportionality constant $\gamma$ so that gives us equation (2)

$$\frac{dR}{dt} = R' \quad = \quad \gamma I. \tag{2}$$

Finally since the change of $I$ is the change of $S$ moving into $I$ and minus the change of $R$ coming from $I$ we have equation (3) and it's equivalent, equation (4):

$$\begin{aligned} \frac{dI}{dt} = I' \quad &= \quad -\frac{dS}{dt} - \frac{dR}{dt} \\ &= \quad -S' - R' \\ &= \quad \lambda I S - \gamma I \qquad (3) \\ &= \quad (\lambda S - \gamma) I. \qquad (4) \end{aligned}$$

Following are some conclusions that can be made regarding this system of equations without solving them. Observe that by assumption $N(t)$ is constant so:

$$\begin{aligned} N(t) \quad &= \quad N(0) = N_0 \\ S(t) + I(t) + R(t) \quad &= \quad N_0 \\ S'(t) + I'(t) + R'(t) \quad &= \quad 0. \end{aligned}$$

Since $S, I$ and $\lambda$ are all positive, equation (1) tells us that $S'(t) < 0$ so,

$$\begin{aligned} S'(t) \quad &< \quad 0 \\ \therefore \quad S(t) \quad &< \quad S_0 \text{ for all } t > 0 \\ \text{So } \lambda S - \gamma \quad &< \quad \lambda S_0 - \gamma \\ (\lambda S - \gamma) I \quad &< \quad (\lambda S_0 - \gamma) I \\ I' \quad &< \quad (\lambda S_0 - \gamma) I. \end{aligned}$$

If $I' < 0$ then the epidemic fizzles out. So using this result with equation (3) from above we have that if $\lambda S_0 - \gamma < 0$ then the epidemic fizzles; this happens when

$$\begin{aligned}
\lambda S_0 - \gamma &< 0 \\
\lambda S_0 &< \gamma \\
S_0 &< \frac{\gamma}{\lambda}.
\end{aligned}$$

For a fixed disease we have no control over the constant $\gamma$. But we have some control over the "constant" $\lambda$: since we assume that $N(t)$ is the constant $N_0$ we have

$$\lambda = \frac{n\alpha}{N_0}$$

where, again, $\alpha$ is a constant related to the disease. But we can decrease $\alpha$ by making it less likely that when a susceptible meets an infectious that there will be a new infection by using masks and social distancing. Assuming we do this to obtain $\alpha'$ a smaller $\alpha$, then since we want $I' < 0$. So we want:

$$\begin{aligned}
S_0 &< \frac{\gamma}{\lambda} \\
S_0 &< \frac{\gamma N_0}{n\alpha'} \\
n\alpha' S_0 &< \gamma N_0 \\
n &< \frac{\gamma N_0}{\alpha' S_0}.
\end{aligned}$$

Assuming $S_0 \approx N_0$ so that $\frac{N_0}{S_0} \approx 1$ this gives us a way to combat an epidemic. Obviously the smaller the value of $n$, the smaller the average number of interactions between members of our population, the faster the epidemic will end. But we must at least have $n < \frac{\gamma}{\alpha'}$ in order to have any hope that the epidemic can be stopped. So the strategy of decreasing $n$ by sequestering enough individuals well below the $n < \frac{\gamma}{\alpha'}$ threshold will eventually cause the epidemic to end.