Math 5000/6000 Test 01

Sept. 27, 2022

Show all your work for each problem, if the work is incomplete or incorrect you may not receive full credit for that problem. If you do scratch work, indicate what is scratch work; no credit will be taken off for errors in the scratch work.

Problem 1. Suppose that a 200 gallon tank is filled with a solution of saltwater that contains 200 g of salt and that saltwater containing 5 g of salt per gallon is entering the tank at a rate of 20 gal/min. Suppose further that the well-mixed solution is leaving the tank at the same rate 20 gal/min.

a.) Find the formula for the amount of salt in the tank as a function of time.

Solution.

$$\begin{aligned}
Q' &= \text{Rate}_{\text{in}} - \text{Rate}_{\text{out}} \\
&= 100 - \frac{Q}{200} \cdot 20 \\
&= 100 - \frac{Q}{10} \\
\frac{Q'}{100 - \frac{Q}{10}} &= 1 \\
\frac{10Q'}{1000 - Q} &= 1 \\
-10\ln(1000 - Q) &= t + C \\
\ln(1000 - Q) &= \frac{-t}{10} + C' \\
1000 - Q &= Ae^{\frac{-t}{10}} \\
Q &= 1000 - Ae^{\frac{-t}{10}}.
\end{aligned}$$

We have $Q(0) = 200$ which gives us $A = 800$. So the solution is

$$Q(t) = 1000 - 800e^{\frac{-t}{10}}.$$

□

b.) What is the limiting concentration of salt?

Solution. $Q \to 1000$ as $t \to \infty$. So the limiting concentration is $\frac{1000}{200} = 5\text{g/gal}$. $\square$

c.) When will the concentration in the tank be within 10% of its limiting value?

Solution.

$$\begin{aligned}
900 &= 1000 - 800e^{\frac{-t}{10}} \\
-100 &= -800e^{\frac{-t}{10}} \\
\frac{1}{8} &= e^{\frac{-t}{10}} \\
\ln\left(\frac{1}{8}\right) &= \frac{-t}{10} \\
-10\ln\left(\frac{1}{8}\right) &= t \\
10\ln 8 &= t \\
20.79441542 min &= t.
\end{aligned}$$

$\square$

Problem 2. Consider the following autonomous differential equation:

$$\frac{dy}{dt} = (1-y)(4-y)(6-y)^2.$$

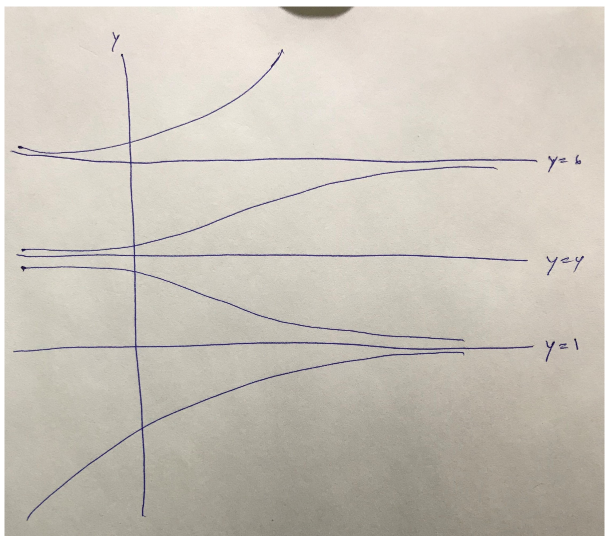
a.) Find the critical points.
b.) Indicate the stability of each critical point as $t \to \infty$.
c.) Sketch at least one graph of the solution in the t-y plane for the regions outside of the critical values; in particular give the limiting value (as $t \to \infty$) for each of the following initial values:

$$\begin{aligned}
(i) &\quad y(0) = 0; \\
(ii) &\quad y(0) = 2; \\
(iii) &\quad y(0) = 5.
\end{aligned}$$

Solution. Parts a & b:

critical point	stability
$y = 1$	stable
$y = 4$	unstable
$y = 6$	semi-stable.

Part c:



$$\begin{aligned} y(0) = 0 &\quad \text{then} \quad y \to 1, \text{as } t \to \infty \\ y(0) = 2 &\quad \text{then} \quad y \to 1, \text{as } t \to \infty \\ y(0) = 5 &\quad \text{then} \quad y \to 6, \text{as } t \to \infty. \end{aligned}$$

□

Problem 3. Find the general solution of the following system of two equations. (Note x' means $\frac{dx}{dt}$ and similarly y' means $\frac{dy}{dt}$.) Also describe the limiting behavior of the general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. The characteristic equation is:

$$\begin{aligned}(1-\lambda)(-6-\lambda)+12 &= \lambda^2+6\lambda-\lambda-6+12\\ &= \lambda^2+5\lambda+6\\ &= (\lambda+2)(\lambda+3).\end{aligned}$$

So the eigenvalues are -2 and -3.

For $\lambda = -2$:

$$\begin{pmatrix} 1-\lambda & -3 \\ 4 & -6-\lambda \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & -3 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

yields $3a - 3b = 0$ or $a = b$ so the eigenvector for $\lambda = -2$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -3$:

$$\begin{pmatrix} 1-\lambda & -3 \\ 4 & -6-\lambda \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

yields $4a - 3b = 0$ or $a = \frac{3}{4}b$ so the eigenvector for $\lambda = -3$ is $\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$ or equivalently $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

So the general solution is:

$$C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{-3t}.$$

□

Problem 4. Find the general solution of the following system of two equations. Also describe the limiting behavior of the general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. The characteristic equation is:

$$\begin{aligned}(-2-\lambda)(-2-\lambda)+4 &= 0\\ \lambda^2+4\lambda+8 &= 0\\ \lambda &= \frac{-4\pm\sqrt{16-4\cdot 8}}{2}\\ &= \frac{-4\pm\sqrt{-16}}{2}\\ &= -2\pm 2i\end{aligned}$$

We'll use the $\lambda = -2+2i$ eigenvalue:

$$\begin{pmatrix} -2-\lambda & 4 \\ -1 & -2-\lambda \end{pmatrix} = \begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix}.$$

To obtain an eigenvector:

$$\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

yields

$$\begin{aligned} -a-2ib &= 0\\ a &= -2ib.\end{aligned}$$

Setting $b=1$ gives us the eigenvector $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$.

So a solution is

$$\begin{aligned}\begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(-2-2i)t} &= e^{-2t}\begin{pmatrix} -2i \\ 1 \end{pmatrix}(\cos(2t)-i\sin(2t))\\ &= e^{-2t}\begin{pmatrix} -2i\cos(2t)-2\sin(2t) \\ \cos(2t)-i\sin(2t) \end{pmatrix}\\ &= e^{-2t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + ie^{-2t}\begin{pmatrix} -2\cos(2t) \\ -\sin(2t) \end{pmatrix}.\end{aligned}$$

So the general solution is:

$$C_1e^{-2t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + C_2e^{-2t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

□

Extra Credit.

Consider the following system of two equations with the parameter α:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & -3 \\ \alpha & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the values of α at which graph of the solution in the phase plane is bounded and does not limit to zero.