Math 5000 Summer 2021
Project 02 Key

The project is due by midnight Friday July 16. Make sure to show your work, if the work is incomplete or incorrect you may not get full credit. As usual, send me a pdf of your work via email with your last name beginning the file name.

A couple of hints:

For the first problem when I designed the problem I used the Euler method to make sure the system was doing what I was trying to do.

For the second problem, explain what modifications you make to the model to answer the questions and explain why you feel these particular modifications best answer the questions asked.

Problem 1. Two competing species (represented by x and y) are preyed upon by a third species(represented by z). Assume that in some common units, that the differential equations that govern their populations is given by:

$$\begin{aligned} x' &= x(2 - x - 0.5y - 0.75z) \\ y' &= y(2.5 - 0.75x - 0.5y - z) \\ z' &= z(-0.25 + 0.5x + 0.5y). \end{aligned}$$

i.) Find the critical point at which all three species coexist (where none have population 0).

Solution. The required critical point is the point that satisfies these equations:

$$\begin{aligned} x + 0.5y + 0.75z &= 2 \\ 0.75x + 0.5y + z &= 2.5 \\ 0.5x + 0.5y &= 0.25. \end{aligned}$$

The solution is

$$x = \frac{1}{5}; y = \frac{3}{10}; z = \frac{11}{5}.$$

□

ii.) Find the equation that the eigenvalue for this critical point satisfies (it will be a cubic equation.)

Solution. The Jacobian is

$$\begin{pmatrix} 2-2x-\frac{1}{2}y-\frac{3}{4}z & -\frac{1}{2}x & -\frac{3}{4}x \\ -\frac{3}{4}y & \frac{5}{2}-\frac{3}{4}x-y-z & -y \\ \frac{1}{2}z & \frac{1}{2}z & -\frac{1}{4}+\frac{1}{2}x+\frac{1}{2}y. \end{pmatrix}$$

Substituting the critical point values gives:

$$\begin{pmatrix} -\frac{4}{20} & -\frac{2}{20} & -\frac{3}{20} \\ \\ -\frac{9}{40} & -\frac{3}{20} & -\frac{6}{20} \\ \\ \frac{22}{20} & \frac{22}{20} & 0 \end{pmatrix} = \frac{1}{8000}\begin{pmatrix} -4 & -2 & -3 \\ \\ -\frac{9}{2} & -3 & -6 \\ \\ 22 & 22 & 0 \end{pmatrix}.$$

The equation that needs to be solved (after multiplying through by a negative) is

$$\hat{\lambda}^3 + 7\hat{\lambda}^2 + 201\hat{\lambda} + 165 \quad = \quad 0$$

where $\hat{\lambda} = 20\lambda$. substituting back $\hat{\lambda} = 20\lambda$ gives us

$$\begin{aligned} \lambda^3 + \frac{7}{20}\lambda^2 + \frac{201}{400}\lambda + \frac{165}{8000} \quad &= \quad 0 \\ \lambda^3 + 0.35\lambda^2 + 0.5025\lambda + 0.020625 \quad &= \quad 0. \end{aligned}$$

Several student solves the cubic (approximately) but you really don't need to do that. (I used Newton's method and got $r = -0.04213$.)

□

iii.) It is possible to determine the stability of this critical point without knowing the value of the critical point. Show that all the real solutions of the equation from part (ii) must be negative.

Solution. If we let

$$f(x) \quad = \quad x^3 + 7x^2 + 201x + 165$$

then we observe that $f(x) > 0$ for $x > 0$. So any root $\hat{\lambda}$ of the cubic must be negative and so $\lambda = \hat{\lambda}/20$ must be negative. □

iv.) Argue that the derivative of the cubic obtained in part ii is never zero: this implies that the cubic is either always increasing or always decreasing.

Solution. Let $g(\lambda)$ be the cubic polynomial of interest; then

$$\begin{aligned} g(\lambda) = \frac{1}{8000} f(20\lambda) &= \lambda^3 + 0.35\lambda^2 + 0.5025\lambda + 0.020625 \\ g'(\lambda) &= 3\lambda^2 + 0.7\lambda + 0.5025. \end{aligned}$$

Using the discriminate we have $\sqrt{b^2 - 4ac} = \sqrt{(0.7)^2 - 4 \cdot 3 \cdot 0.5025}$ which is imaginary. So function g is always increasing (or decreasing in the case where one did not multiply out the negative) hence g has one root and two imaginary roots. □

v.) It turns out that the other roots of the equation of part (ii) are complex with negative real part [extra credit if you can show this]. Argue that this fact together with the other findings from above, imply that the critical point is a stable attractive fixed point.

Solution. If $\lambda = -r$ is the real root of g and the complex roots of g are $-a \pm bi$, then the general solution of the linearized equation near the critical point found above (with $r, a > 0$) will be the form:

$$X = c_1 \overrightarrow{v_1} e^{-rt} + c_2 \overrightarrow{v_2} e^{(-a+bi)t} + c_3 \overrightarrow{v_3} e^{(-a-bi)t}.$$

This means that the critical point is an attractive spiral point or (in the case that $c_1 = c_2 = 0$) it is an attractive node. □

Extra credit.

Solution. We know that there is only one real root. We'd like to know that the real part of the complex root is negative. For a real function of the form $h(x) = x^3 + \alpha x^2 + \beta x + \gamma$, the root are r and $a \pm bi$. Then the α term is given by $\alpha = -r - 2a$. [This is because $h(x) = (x-r)(x-a-bi)(x-a+bi)$.] So this gives us:

$$\begin{aligned} -r - 2a &= \alpha \\ 2a &= -r - \alpha. \end{aligned}$$

In order for a to be negative we need

$$\begin{aligned} -r - 2a &= \alpha \\ 2a = -r - \alpha &< 0 \\ -\alpha &< r. \end{aligned}$$

This is easy to check! Since, in this case $r < 0$ (and $h(0) > 0$), then calculating $h(-\alpha)$ so see if it's negative means that r is between $-\alpha$ and 0 which is what we needed. In our case $\alpha = 0.35$. [I calculated $h(-\alpha) = -0.15525$.] So a is negative. This means that, as shown above, that the critical point is an attractive fixed point; from most initial conditions, the solutions spirals in toward the point. □

Problem 2. Use the Euler method to model the solution to the SIR model of an epidemic. Suppose that an epidemic is invading a state with a population of about 4,000,000. Suppose that this happened when 5 infected individuals arrive in the state at time $t = 0$. Assume time is given in days and the parameters of the SIR model have the following values:

$$\begin{aligned} n &= 20 \\ \alpha &= 0.055 \\ \gamma &= 0.8. \end{aligned}$$

Suppose a death rate of 2.1% of the infected individuals.

i.) If nothing is done, will the number of people who catch the disease every reach $1,000,000$? If yes, determine approximately when.

Solution. Yes depending on the h value used you should come up with a value of $t = 39 - 44$:

h value	t in days
0.05	39.35
0.1	39.5
0.2	40
1.0	43.5

□

ii.) Assume nothing is done for sixty days. How many deaths would be predicted by the model?

Solution. About $40,984 - 42,284$.

□

iii.) Suppose that a vaccination program is implemented after 30 days. The vaccination is 90% effective. Assume that the population, through some miraculous health care workers, is completely vaccinated on day 31 - so modify the model by reducing the α value to $\alpha = 0.0055$. How many deaths would there now be by day 60? How many deaths does the model predict that the vaccination program would prevent by the sixtieth day?

Solution. If vaccination is implemented at $t = 30$ days, there would only be about 2885 deaths at day $t = 60$. So about $38,099 - 39,399$ lives would be saved.

□

iv.) Suppose that half the population elects not to be vaccinated. (Assume that these members of the population only interact among themselves so that the value of n can be kept at 20.) How many additional death would this decision cause after 60 days above the number of deaths predicted in part (iii)?

Solution. I calculated this by assuming the same model but dividing the S, I and R numbers at day 30 by 2. Alternately one could use half the original initial conditions: This gives If vaccination is implemented at $t = 30$ days. Depending on your h values, this gives about $17,685$ additional death above the 2885 calculated above. Just dividing the answer in part (iii) by 2 overestimates this quantity.

□

Problem 3. An ecologist is working in an ecosystem that includes a prey (with population x) and predator (with population y). He is modeling the relationship of between the two species with populations x and y (in appropriate units) with the following system of equations:

$$\begin{aligned} x' &= x(2 - \alpha x - y) \\ y' &= y(-\frac{1}{3} + \frac{x}{4}) \end{aligned}$$

where α is a positive constant that he wishes to determine.

i.) In terms of α find the critical point at which both population are non-zero.

Solution. This happens when:

$$x = \frac{4}{3}; y = 2 - \frac{4}{3}\alpha.$$

□

ii.) Determine the range of possible values for α that causes the critical point of part (i) to be stable.

Solution. The Jacobian is

$$\begin{pmatrix} 2 - 2\alpha x - y & -x \\ \frac{y}{4} & -\frac{1}{3} + \frac{x}{4} \end{pmatrix}.$$

Substituting the critical point values gives:

$$\begin{pmatrix} -\frac{4\alpha}{3} & -\frac{4}{3} \\ \\ \frac{1}{2} - \frac{\alpha}{3} & 0 \end{pmatrix}.$$

The equation for the eigenvalue λ is

$$\lambda^2 + \frac{4}{3}\alpha\lambda + \frac{2}{3} - \frac{4}{9}\alpha = 0.$$

$$\lambda = \frac{1}{2}\Big(- \frac{4}{3}\alpha \pm \sqrt{\frac{16}{9}\alpha^2 - \frac{8}{3} + \frac{16}{9}\alpha}\Big).$$

This quantity will be negative (or imaginary) as long as the square root is less than $\frac{4}{3}\alpha$. This happens when $\alpha < 1.5$. This is consistent with the y value of critcal point being positive:

$$\begin{aligned} y &= 2 - \frac{4}{3}\alpha \\ 0 &< y \\ 0 &< 2 - \frac{4}{3}\alpha \\ \alpha &< \frac{3}{2}. \end{aligned}$$

In fact this turns out to be equvialent to the range obtained above. $\square$

iii.) There is a cutoff value for α where the critical point of part (i) switches from an attractive node to a spiral point. Determine the value of α at which this occurs.

Solution. The occurs when the discriminant of the equation for λ is zero. That equation is

$$\begin{aligned} \frac{16}{9}\alpha^2 - \frac{8}{3} + \frac{16}{9}\alpha &= 0 \\ \alpha^2 + \alpha - \frac{3}{2} &= \\ \alpha &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{7}. \end{aligned}$$

Since α is required to be positive this yields $\alpha = -\frac{1}{2} + \frac{1}{2}\sqrt{7} \approx 0.822876$ as the cut-off value. $\square$