Math 5000 Summer 2021
Test 02, Key

The test is due Monday July 26 before midnight. The test is open book and open notes - this includes my notes on the website. You may not receive any other outside assistance and may not discuss the test with anyone. **Please affirm at the beginning of your hand-in work that you have abided by these conditions.** (If you do not do so, you may be penalized.)

Email to me as an attachment your solutions to the problems as a pdf file with the file name beginning with your last name: e.g. LastName_test02.pdf. (If you do not do so, you may be penalized.)

Show all your work, you may not get full credit if the solution is incomplete or incorrectly done.

Problem 1. An ecologist is studying the relationship between a prey (with population x) and a predator (with population y) in an ecosystem. He found that there is a parameter α so that, in appropriate units, the following system of differential equation models the populations of the prey and predator species.

$$\begin{aligned} x' &= x(2-2y-\alpha x) \\ y' &= y\Big(2x-\frac{1}{2}\Big) \end{aligned}$$

i.) In terms of α, find the critical point at which both population are non-zero.

Solution. The critical point that gives non-zero values for x and y satisfy:

$$\begin{aligned} 2-2y-\alpha x) &= 0 \\ 2x-\frac{1}{2} &= 0. \end{aligned}$$

This yields

$$\begin{aligned} x &= \frac{1}{2} \\ y &= 1-\frac{\alpha}{8}. \end{aligned}$$

□

ii.) He suspects that, no matter what value α has as long as the y coordinate of critical point obtained in (i) is positive, then the critical point is an attractive fixed point. Show that his suspicion is correct.

Solution. The Jacobian is

$$J(x,y) = \begin{pmatrix} 2 - 2y - 2\alpha x & -2x \\ 2y & 2x - \frac{1}{2} \end{pmatrix}.$$

So at the critical point

$$J = \begin{pmatrix} -\frac{\alpha}{4} & -\frac{1}{2} \\ 2 - \frac{\alpha}{4} & 0 \end{pmatrix}.$$

For the eigenvalue we set the determinant of the following matrix equal to 0.

$$J = \begin{pmatrix} -\frac{\alpha}{4} - \lambda & -\frac{1}{2} \\ 2 - \frac{\alpha}{4} & -\lambda \end{pmatrix}.$$

This gives us

$$\begin{aligned} \lambda\Big(\lambda + \frac{\alpha}{4}\Big) + 1 - \frac{\alpha}{8} &= 0 \\ \lambda^2 + \frac{\alpha}{4}\lambda + 1 - \frac{\alpha}{8} &= 0 \end{aligned}$$

$$\lambda = \frac{1}{2}\Big(-\frac{\alpha}{4} \pm \sqrt{\frac{\alpha^2}{16} - 4\Big(1 - \frac{\alpha}{8}\Big)}\ \Big).$$

In order to have an attractive fixed point the real part of λ must be negative. So $\alpha > 0$ And we need the square root to be imaginary or less than $\frac{\alpha}{4}$. So

$$\sqrt{\frac{\alpha^2}{16} - 4\Big(1 - \frac{\alpha}{8}\Big)} < \frac{\alpha}{4}.$$

Which gives us

$$\begin{aligned} 1 - \frac{\alpha}{8} &> 0 \\ 8 &> \alpha. \end{aligned}$$

We observe that the y coordinate of the critical point must be positive when

$$\begin{aligned} 1 - \frac{\alpha}{8} &> 0 \\ 8 &> \alpha. \end{aligned}$$

Which confirms the ecologist's suspicion. $\square$

iii.) There is a cutoff value for α where the critical point of part (i) switches from an attractive node to a spiral point. Determine the value of α at which this occurs.

Solution. This occurs when the discriminant of λ above is zero:

$$\begin{aligned} \frac{\alpha^2}{16} - 4\left(1 - \frac{\alpha}{8}\right) &= 0 \\ \frac{\alpha^2}{16} + \frac{\alpha}{2} - 4 &= 0 \\ \alpha^2 + 8\alpha - 64 &= 0 \\ \alpha &= -4 \pm 4\sqrt{5}. \end{aligned}$$

We take the positive root since α must be positive. $\square$

Problem 2. Our ecologist feels that the system of equations of problem 1 can be improved by including a term for the competition of the members of the prey species among themselves. For the $\alpha = 2$ value he came up with the following system of equations.

$$\begin{aligned} x' &= x(2 - 2y - 2x) \\ y' &= y\Big(2x - \frac{1}{2} - y\Big). \end{aligned}$$

Find all the critical points of this system and indicate the stability of each of them.

Solution. For $\alpha = 2$ the critical points are
$(0,0); (0, -\frac{1}{2}); (1, 0); (\frac{1}{2}, \frac{1}{2})$.
The Jacobian is

$$J(x,y) = \begin{pmatrix} 2 - 2y - 4x & -2x \\ 2y & 2x - \frac{1}{2} - 2y \end{pmatrix}.$$

At $(0,0)$

$$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

This gives

$$\begin{aligned} (2 - \lambda)\Big(-\frac{1}{2} - \lambda\Big) &= 0 \\ \lambda &= 2, -\frac{1}{2}. \end{aligned}$$

Saddle point, unstable.

At $(0, -\frac{1}{2})$

$$J(0, -\frac{1}{2}) = \begin{pmatrix} 3 & 0 \\ -1 & \frac{1}{2} \end{pmatrix}.$$

This gives

$$\begin{aligned} (3 - \lambda)\Big(\frac{1}{2} - \lambda\Big) &= 0 \\ \lambda &= 3, \frac{1}{2}. \end{aligned}$$

Repelling fixed point, unstable.

At $(1, 0)$

$$J(1,0) = \begin{pmatrix} -2 & -2 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

This gives

$$\begin{aligned} (-2-\lambda)\Big(\frac{3}{2}-\lambda\Big) &= 0 \\ \lambda &= -2, \frac{3}{2}. \end{aligned}$$

Saddle point, unstable.

At $(\frac{1}{2}, \frac{1}{2})$

$$J\Big(\frac{1}{2}, \frac{1}{2}\Big) = \begin{pmatrix} -1 & -1 \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

This gives

$$\begin{aligned} (-1-\lambda)\Big(-\frac{1}{2}-\lambda\Big)+1 &= 0 \\ \lambda^2 + \frac{3}{2}\lambda + \frac{3}{2} &= 0 \\ 2\lambda^2 + 3\lambda + 3 &= 0 \\ \lambda &= (-3 \pm \sqrt{-15})/4. \end{aligned}$$

Attractive spiral point, stable. □

Problem 3. Find all the values of λ for which the following equation with the indicated boundary condition has a non-zero solution.

$$y'' + 10y' + \lambda y = 0 \quad \text{for } x \in [0, L] \quad y'(0) = 0; y'(L) = 0.$$

[Note: the values of λ will be in terms of L.]

Solution. To solve we assume y is in the form $y = e^{rt}$. The polynomial equation for r is

$$
\begin{aligned}
r^2 + 10r + \lambda &= 0 \\
r &= -5 \pm \sqrt{25-\lambda}.
\end{aligned}
$$

The root r will be real if $25 \geq \lambda$.

Case 1: $25 > \lambda$. Then there are two roots $r_1 = a$, $r_2 = a + b$ (I express them this way to make the algebra easier. The general solution is

$$
\begin{aligned}
y &= c_1 e^{at} + c_2 e^{(a+b)t} \\
y' &= d_1 e^{at} + d_2 e^{(a+b)t} \\
&= e^{at}(d_1 + d_2 e^{bt}) \\
y(0) = 0 &= d_1 + d_2.
\end{aligned}
$$

Therefore $d_1 = -d_2$. So

$$
\begin{aligned}
y' &= d_1 e^{at} - d_1 e^{at} e^{bt} \\
y'(L) = 0 &= d_1 e^{aL} - d_1 e^{aL} e^{bL} \\
d_1 &= d_1 e^{bL}.
\end{aligned}
$$

Therefore $L = 0$ but this is a contradiction. So there are no solutions if $25 > \lambda$.

[Now, I inadvertently assumed $d_1 \neq 0$. This actually gives us a solution: When $\lambda = 0$, $y = 1$ is a solution.]

Case 2: $\lambda = 25$. Then $r = -5$ is the only root of the auxiliary polynomial (i.e. "repeated root") and the general solution is

$$
y = c_1 e^{-5t} + c_2 t e^{-5t}.
$$

So:

$$
\begin{aligned}
y' &= (-5c_1 + c_2)e^{-5t} - 5c_2 t e^{-5t} \\
y'(0) &= 0 = -5c_1 + c_2 \Rightarrow c_2 = 5c_1 \\
y'(L) &= 0 = -5c_2 L e^{-5L} \\
&\Rightarrow c_2 = 0 \text{ and then } c_1 = 0.
\end{aligned}
$$

So there is no non-zero solution in this case.

Case 3: $25 < \lambda$. Then r is in the form $r = -5 + bi$ where $b = \sqrt{\lambda - 25}$; the general solution is

$$y = c_1 e^{-5t} \cos bt + c_2 e^{-5t} \sin(bt).$$

Then y' takes on the form

$$y' = d_1 e^{-5t} \cos bt + d_2 e^{-5t} \sin(bt).$$

Where the d's are related to the c's in such a way that the $d_1 = 0, d_2 = 0$ implies $c_1 = 0, c_2 = 0$. [The relevant equations, according to my calculations are: $-5c_1 + bc_2 = d_1$ and $-5c_2 - bc_1 = d_2$.]

Then using the boundary condition

$$\begin{aligned} y'(0) = 0 &= d_1 \\ y'(L) = 0 &= d_2 e^{-5L} \sin(bL). \end{aligned}$$

The only way for d_2 to not be zero is if $bL = n\pi$, this happens when

$$\begin{aligned} bL &= n\pi \\ b &= \frac{n\pi}{L} \\ \sqrt{\lambda - 25} &= \frac{n\pi}{L} \\ \lambda - 25 &= \frac{n^2\pi^2}{L^2} \\ \lambda &= 25 + \frac{n^2\pi^2}{L^2}. \end{aligned}$$

□

Problem 4. Find the Fourier series for the following function defined on $[-2, 2]$.

$$f(x) = \begin{cases} x + 2 & \text{if } -2 \le x < 0 \\ 2 - x & \text{if } 0 \le x \le 2. \end{cases}$$

Solution. The integration gives $a_0 = 2$ and (by the symmetry since f is an even function):

$$f(x) = 2\frac{1}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx$$

$$a_n = (2-x)\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)\Big|_0^2.$$

$$f(x) \quad = \quad 1 + \sum_{n \text{ odd}} \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right).$$

□

Problem 5. Suppose that $u(x,t)$ is the solution of the heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

with $\alpha = 3$ and with the following boundary condition over the interval $[0, L] = [0, 2]$.

$$\begin{aligned} u(x,0) &= x^2 \text{ for } x \in [0,2] \\ u(0,t) &= 0 \text{ for } t \geq 0 \\ u(2,t) &= 0 \text{ for } t \geq 0. \end{aligned}$$

i.) **Just set up the integrals**, with correct limits of integration, that gives the constants that are the coefficients of the series solution of wave equation.

Solution. The eigenvalue λ will be

$$\lambda \quad = \quad \frac{n^2\pi^2}{4}.$$

The required coefficients are

$$c_n \quad = \quad \int_0^2 x^2 \sin\left(\frac{n\pi}{2}x\right) dx, \quad n = 1, 2, 3, \ldots.$$

□

ii.) In terms of the constants determined in (i) above, give the complete solution of the heat equation with these boundary conditions. [Extra credit: evaluate the integrals and obtain the series solution.]

Solution. The series solution of the heat equation, in terms of the constants calculated above, will be,

$$u(x,t) \quad = \quad \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2}x\right) e^{\frac{-n^2\pi^2 9}{4}t}.$$

$\square$