Math 5000/6000, Summer 2021, Dr. Smith
Test 01, Key

Instructions: The test is due by midnight Monday June 21. The test is open book and open notes; this includes my notes on the website. You may not receive any other outside assistance and may not discuss the quiz with anyone. Please affirm at the beginning of your hand-in work that you have abided by these conditions.

Email to me as an attachment your solutions to the problems as a pdf file with the file name beginning with your last name: e.g. smithxyztest01.pdf.

Problem instructions:
Construct models to analyze the following situations. In each case:

i.) set up the differential equation that models the problem;
ii.) solve the equation and write up details of the solution;
iii.) use it to answer the questions asked;
iv.) determine the limiting behavior of the modeled system.

Problem 1. A lake contains 2gm/gal of a pollutant. A stream that feeds into the lake at a rate of 20,000 gal/day contains only 0.1 gm/gal of the pollutant and the lake drains into a river at the same rate. Assume instantaneous mixing. It takes (exactly) one year for the pollutant concentration of the lake to decrease to 10% of it's original level. Question: how large is the lake (volume in gallons.)

solution. Let Q denote the quantity of the pollutant in the lake; let V denote the volume of the late. Then

$$\begin{aligned} Q'(t) &= \text{Rate In - Rate Out} \\ &= 2,000\text{gm/day} - \frac{Q}{V}20,000\text{gm/day} \\ \frac{Q'(t)}{0.1 - \frac{Q}{V}} &= 1. \end{aligned}$$

This yields

$$Q(t) = 0.1V + Ae^{-\frac{20,000t}{V}}$$

for some constant A. The boundary condition is $\frac{Q(0)}{V} = 2\text{gm/day}$ which gives us $A = 1.9V$, so

$$\begin{aligned} Q(t) &= 0.1V + 1.9Ve^{-\frac{20,000t}{V}} \\ \frac{Q(t)}{V} &= 0.1 + 1.9e^{-\frac{20,000t}{V}}. \end{aligned}$$

Since $Q(365.25)/V = 0.2$ this give us,

$$\begin{aligned} 0.2 &= 0.1 + 1.9e^{-\frac{20,000t}{V}} \\ 0.1 &= 1.9e^{-\frac{20,000t}{V}} \\ V &= \frac{20,000 \cdot 365.25}{\ln 19} \text{ gal.} \\ &= 2,480,948 \text{ gal.} \end{aligned}$$

For the limiting behaviour we see that as $t \to \infty$ we have $\frac{Q}{V} \to 0.1$.

□

Problem 2. During the pandemic a baker discovered that she was unable to obtain yeast (due to the huge number of self-isolated people deciding to bake.) So she decided to make up a batch of sour-dough. This entails starting a "natural wild" yeast culture and "feeding" it with flour and water once a day. Her bread baking book says that to feed the culture she must throw out 80% of her culture and replace it with flour and water. Let's suppose that the yeast population follows a logistic model: $y' = ay(1 - y)$ where y represents the proportion of yeast that living on the available flour at time t. She concludes that if the yeast starts with 20% of the maximum then it takes it one day to eat up most of the flour. Assume that starting with a 20% yeast solution it takes it a day to expand to 90% of capacity.

i.) Plot the phase diagram of possible solutions and discuss the stability of each fixed solution. Indicate which region best models this experiment.

Solution.

$$\begin{aligned} y = 0 &: \text{ unstable} \\ y = 1 &: \text{ stable.} \end{aligned}$$

□

ii.) Solve the equation given the initial value $y(0) = 0.2$.

Solution.

$$y = \frac{e^{at}}{4 + e^{at}} = \frac{1}{4e^{-at} + 1}.$$

□

iii.) Calculate a based on the conditions stated in the problem.

Solution.

$$\begin{aligned} a &= \ln(36) \approx 3.5835 \text{ (time measured in days)} \\ a &= \ln(36)/24 \approx 0.149313 \text{ (time measured in hours).} \end{aligned}$$

□

iv.) If the baker forgot to "feed" her culture after one day, how much has the yeast filled its capacity after two days.

Solution.

$$y(2) \approx 0.996923.$$

The amount is independent of the time units used. □

Problem 3. A geologists is designing a seismograph. The instrument's design is based on the mechanical vibration of the stylus as a weight on a spring. The stylus moves when there is an earthquake and the stylus is given a sudden initial velocity. The mass of the stylus is 0.1 kg, the damping factor is 0.5 units and Hooke's constant for the spring engaging the stylus is 0.4. Before an earthquake happens the stylus is still. Suppose an earthquake occurs so that the stylus starts from an initial distance of 0 from it's equilibrium and it is given a sudden initial velocity of v m/sec.

i.) Set up the equation of motion for the stylus and solve the equation in terms of the initial velocity v.

Solution. Equation of motion:

$$0.1u'' + 0.5u' + 0.4u = 0.$$

or

$$u'' + 5u' + 4u = 0.$$

For the given initial value:

$$u(t) = \frac{s}{3}e^{-t} - \frac{s}{3}e^{-4t}.$$

□

ii.) Calculate when the stylus reaches its maximum displacement, in terms of v. Then calculate that displacement.

Solution.

$$t_{\text{max}} = \frac{\ln(4)}{3} \approx 0.462098.$$

□

iii.) Is it possible to determine the initial velocity in terms of maximum displacement of the stylus? If no, say why; if yes calculate the initial velocity in terms of the distance (calculated in part ii) that the stylus traveled. (Note this distance is easily determined by measuring the peak that the stylus records on the paper; since the initial velocity of the stylus is a measure of the magnitude of the earthquake at the seismograph's location, this could be a potential way to measure the magnitude of an earthquake.)

Solution. Let d denote the maximum displacement.

$$\begin{aligned} d = u(t_{\text{max}}) &\approx s(0.15749) \\ s &\approx 6.349609d. \end{aligned}$$

□

Problem 4. An epidemic is raging in a country. At some point during the epidemic, the epidemiologists recommended a social distancing policy. They used the following system of differential equations to model the implementation of this policy where α and γ are positive constants and γ represents a factor due to the social distancing policy (related to the number of people interacting). The variable u (in cases per 100,000) represents the number of people that are infected at time t (measured in fortnights).

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -\alpha & -4-\gamma \\ -1 & -\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The constant α for this epidemic cannot be affected by behavioral changes, for this particular disease it has a value of 5 units.

i.) Determine the range of γ for which the epidemic will asymptotically limit to zero infections.

Solution.

$$0 < \gamma < 21.$$

□

ii.) Consider the value of γ which will end the epidemic quickest, approximate how long it will take for the number of people ill to decrease to 10% of the value when this social distancing policy is implemented. Indicate why you made the approximation that you did.

Solution. Although we are given $\gamma > 0$ using $\gamma = 0$ is a reasonable choice for ending the epidemic quickest since we want an estimate. At $\gamma = 0$ the general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-7t}.$$

We don't know the initial number of infections when the policy was implemented. However we want to estimate when it goes down to 0.1. Since u is the variable that counts the number of infections we have:

$$\begin{aligned} u &= -2c_1 e^{-3t} + 2c_2 e^{-7t} \\ u(0) &= -2c_1 + 2c_2. \end{aligned}$$

Since e^{-3t} dominates as $t \to \infty$ it follows that $c_1 < 0$. And as long as $t > 0$ we have

$$e^{-7t} < e^{-3t}.$$

Since we just want a reasonable estimate, we will ignore the e^{-7t} term and we want to know when

$$\begin{aligned} e^{-3t} &= 0.1 \\ t &\approx 0.767528 \end{aligned}$$

□

iii.) At the value of $\gamma = 12$, approximate how long it will take for the number of people ill to decrease to 1% of the value when the social distancing policy is implemented with this value of γ. Indicate why you made the approximation that you did.

Solution. In a similar way as (ii) above, the general solution for $\gamma = 12$ is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-9t}.$$

Again as above, since we just want a reasonable estimate, we will ignore the e^{-9t} term and we want to know when

$$\begin{aligned} e^{-t} &= 0.01 \\ t &\approx \ln(100) = 4.605 \end{aligned}$$

□

This is about three times as long.

Problem 5. Consider the following system of differential equations:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -4 & \alpha \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

i.) Determine the range of α so that all solutions to the system will limit to 0 as $t \to \infty$.

solution. The eigenvalue is given by:

$$\begin{aligned} \lambda^2 + 4\lambda - 2\alpha &= 0, \\ \lambda &= -2 \pm \sqrt{4+2\alpha}. \end{aligned}$$

So the general solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1[v_1]e^{(-2+\sqrt{4+2\alpha})t} + c_2[v_2]e^{(-2-\sqrt{4+2\alpha})t}$$

where $[v_1]$ and $[v_2]$ are the eigenvectors. Observe that if $4 + 2\alpha < 0$ then the imaginary roots lead to sines and cosines and these portions of the solutions are multiplied by e^{-2t} which limits to 0. In order to find all solutions to limit to zero we need $-2 + \sqrt{4+2\alpha} < 0$ or the solutions to be complex; this happens when $\alpha < 0$. □

ii.) Find the general solution for $\alpha = 20$.

solution. The eigenvalue is given by:

$$\begin{aligned} \lambda^2 + 4\lambda - 40 &= 0, \\ \lambda &= -2 \pm \sqrt{4+40} \\ &= -2 \pm 2\sqrt{11}. \end{aligned}$$

So the general solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 1-\sqrt{11} \\ -1 \end{bmatrix} e^{(-2+2\sqrt{11})t} + c_2 \begin{bmatrix} 1+\sqrt{11} \\ -1 \end{bmatrix}]e^{(-2-2\sqrt{11})t}$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 10 \\ 1+\sqrt{11} \end{bmatrix} e^{(-2+2\sqrt{11})t} + c_2 \begin{bmatrix} 10 \\ 1-\sqrt{11} \end{bmatrix}]e^{(-2-2\sqrt{11})t}$$

□

iii.) Find the general solution for $\alpha = 2$.

solution. The eigenvalue is given by:

$$\begin{aligned} \lambda^2 + 4\lambda - 4 &= 0, \\ \lambda &= -2 \pm \sqrt{8} \\ &= -2 \pm 2\sqrt{2}. \end{aligned}$$

So the general solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 1-\sqrt{2} \\ -1 \end{bmatrix} e^{(-2+2\sqrt{2})t} + c_2 \begin{bmatrix} 1+\sqrt{2} \\ -1 \end{bmatrix}]e^{(-2-2\sqrt{2})t}$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix} e^{(-2+2\sqrt{2})t} + c_2 \begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix}]e^{(-2-2\sqrt{2})t}$$

□

iii.) Find the general solution for α at the endpoint[s] of the range of α .

solution. The endpoint of the range of α is 0. So the eigenvalue is given by:

$$\begin{aligned} \lambda^2 + 4\lambda &= 0, \\ \lambda(\lambda + 4) &= 0 \\ \lambda &= -4, 0. \end{aligned}$$

So the general solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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□