Math 5000/6000 Short Project.
Key

Exercise 1.

Consider the following equation that models a harmonic oscillator:

$$16x'' + \gamma x' + 100 = 0.$$

The parameter γ measures the dampening of the oscillator. Suppose that the initial conditions are $x(0) = 0$ and $x'(0) = v_0$.

a.) Determine the smallest value for γ such that the oscillator does not return to equilibrium after it is set in motion at $t = 0$. (But only approaches it asymptotically.)

Solution. The equation yields the following auxiliary equation:

$$\begin{aligned} 16r^2 + \gamma r + 100 &= 0; \\ r &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4 \cdot 16 \cdot 100}}{2 \cdot 16}. \end{aligned}$$

The solution will oscillate around the equilibrium as long as the discriminant is side the radical is negative. We want the first value of γ the makes the discriminant zero:

$$\begin{aligned} \gamma^2 - 4 \cdot 16 \cdot 100 &= 0; \\ \gamma &= 2 \cdot 4 \cdot 10 = 80. \end{aligned}$$

□

b.) In terms of the initial velocity, determine the time at which it attains it's maximum displacement and determine the magnitude of that displacement.

Solution. Since the discriminant is zero the general solution will be:

$$\begin{aligned} x &= c_1 t e^{-\frac{\gamma}{32} t} + c_2 e^{-\frac{\gamma}{32} t} \\ &= c_1 t e^{-\frac{5}{2} t} + c_2 e^{-\frac{5}{2} t}. \end{aligned}$$

For the given boundary condition we have $c_1 = v_0, c_2 = 0$.

To find the maximum we set the derivative equal to zero.

$$\begin{aligned}
x(t) &= v_0 t e^{-\frac{\gamma}{32}t} \\
&= v_0 t e^{-\frac{5}{2}t} \\
x'(t) &= v_0 e^{-\frac{5}{2}t} - \frac{5}{2} v_0 t e^{-\frac{5}{2}t} \\
0 &= v_0 \left(1 - \frac{5}{2}t\right) e^{-\frac{5}{2}t} \\
t &= \frac{2}{5}.
\end{aligned}$$

Observe that the time is independent of the initial velocity. The displacement at time $t = \frac{2}{5}$ is:

$$x\left(\frac{2}{5}\right) = v_0 \frac{2}{5} e^{-1}.$$

□

Exercise 2.

A tank is filled with pure water when, at time $t = 0$, a 5 gm/liter solution of salt water starts flowing into a tank at a rate of 25 liters/min. The salt water continues to flow into the tank at this rate and the well-mixed solution leaves the tank at the same rate.

a.) Express the concentration of salt in the tank at time $t > 0$ in terms of the tank's volume V.

Solution. Let $Q(t)$ denote the amount in grams of salt in the tank. The concentration then will be $\frac{Q(t)}{V}$. The equation that models this problem is:

$$\begin{aligned}
Q' &= \text{rate in } - \text{rate out} \\
&= 125 - \frac{Q}{V} \cdot 25.
\end{aligned}$$

Using the separation of variables technique we have,

$$\begin{aligned}
\frac{Q'}{125 - \frac{Q}{V} \cdot 25} &= 1 \\
-\frac{V}{25} \ln(125 - \frac{Q}{V} 25) &= t + c \\
\ln(125 - \frac{Q}{V} 25) &= -\frac{25}{V} t + c' \\
125 - \frac{Q}{V} 25 &= A e^{-\frac{25}{V}} t.
\end{aligned}$$

The initial condition gives us $A = 125$. So,

$$Q = 5V - 5Ve^{-\frac{25}{V}} t.$$

The concentration then at time t is

$$C(t) = \frac{Q}{V} = 5 - 5e^{-\frac{25}{V}} t.$$

□

b.) If the concentration of the salt in the tank after 20 minutes is 2 gm/liter, what is the volume of the tank?

Solution. The concentration after 20 minutes will be

$$C(20) = 5 - 5e^{-\frac{500}{V}}.$$

Solving for V:

$$V = \frac{-500}{\ln(0.6)} = 978.808 \text{ liters.}$$

□

c.) How long will it take for the concentration of salt in the tank to be 4 gm/liter.

Solution. Solving for t when C(t)=4 grams/liter gives

$$t = -V \frac{\ln(0.2)}{25} = 63.0132 \text{ minutes.}$$

□

Exercise 3.

Consider the following system of differential equations:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Find two linearly independent solutions to this system of equations and find the general solution.

Solution. To find the eigenvalue we need the determinant of the following matrix to be 0.

$$\begin{bmatrix} 1-\lambda & -2 \\ 2 & -5-\lambda \end{bmatrix}.$$

This gives us the equation

$$\begin{aligned} \lambda^2 + 4\lambda - 1 &= 0 \\ \lambda &= -2 \pm \sqrt{5}. \end{aligned}$$

To calculate the eigenvectors for $\lambda = -2 + \sqrt{5}$ we have

$$\begin{bmatrix} 3-\sqrt{5} & -2 \\ 2 & -3-\sqrt{5} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

$$(3-\sqrt{5})a - 2b = 0.$$

Letting $a = 2$ gives $b = 3 - \sqrt{5}$.

Then for $\lambda = -2 - \sqrt{5}$ we have

$$\begin{bmatrix} 3+\sqrt{5} & -2 \\ 2 & -3+\sqrt{5} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

$$(3+\sqrt{5})a - 2b = 0.$$

Letting $a = 2$ gives $b = 3 + \sqrt{5}$.

The general solution is:

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3-\sqrt{5} \end{bmatrix} e^{(-2+\sqrt{5})t} + c_2 \begin{bmatrix} 2 \\ 3+\sqrt{5} \end{bmatrix} e^{(-2-\sqrt{5})t}$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 3+\sqrt{5} \\ 2 \end{bmatrix} e^{(-2+\sqrt{5})t} + c_2 \begin{bmatrix} 3-\sqrt{5} \\ 2 \end{bmatrix} e^{(-2-\sqrt{5})t}$$

$\square$