SIR Model of Infectious Diseases.

We suppose that the have a population S of susceptible hosts for a disease, a population I of infectious host and a population R of recovered hosts. Roughly the situation may be flow charted as in the figure below:

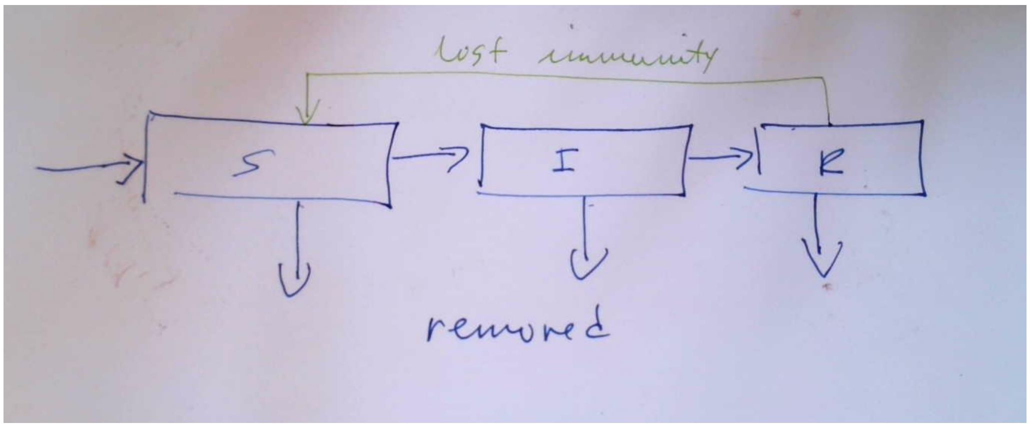

FIGURE 1. SIR Model

We have entrances to the group of susceptibles through births and immigrations and removals through deaths and emigration. Let $N(t)$ denote the total number of hosts:

$$N(t) \quad = \quad S(t) + I(t) + R(t).$$

Simplified SIR Model 01:

For our fist simplified model we will assume:

- No births (or equivalently, birth rates = death rates);
- neither immigration nor emigration;
- removed = 0;
- no loss of immunity.

This simplifies the flow chart to the following:

$$\boxed{S = \text{susceptible}} \longrightarrow \boxed{I = \text{Infective}} \longrightarrow \boxed{R = \text{Recovered}}$$

These assumptions imply $N(t)$ is constant, $N(t) = N(0) = N_0$. Each of the variables S, I, R is a function of time t where, in our cases, t will be in days.

For some quantity A let ΔA denote the change of quantity A from day m at midnight to day $m+1$ at midnight; so ΔA denotes the "(average) change of quantity A per day".

$$\begin{aligned}\Delta S &= -\text{new infections}\\ \Delta I &= \text{new infections} - \text{new recovered}\\ \Delta R &= \text{newly recovered.}\end{aligned}$$

We have:

$$\Delta I = -\Delta S - \Delta R.$$

To set up the differential equations we consider an encounter between a susceptible and an infectious. We assume that there is a constant probability α of infection with each encounter. So assume an average of n encounters per day between members of the population; a particular infectious individual will infect $n\alpha\frac{S}{N_0}$ individuals per day. (Note, there is the possibility of two infectious encountering the same susceptible, this is on the order of $(\frac{n}{N_0})^2$ and for large populations $n << N_0$ this quantity, along with higher order multiple encounters is negligible.) So the total number of individuals infected by the infectious group per day is

$$n\alpha\frac{S}{N_0}I.$$

Let $\lambda = \frac{n\alpha}{N_0}$. This gives us equation (1):

$$\frac{dS}{dt} = S' = -\lambda IS. \tag{1}$$

Assume that infectious individuals become cured at a constant rate proportional to the number of infectious individuals; assume this happens with proportionality constant γ so that gives us equation (2)

$$\frac{dR}{dt} = R' = \gamma I. \tag{2}$$

Finally since the change of I is the change of S moving into I and minus the change of R coming from I we have equation (3) and it's equivalent, equation (4):

$$\begin{aligned}\frac{dI}{dt} = I' &= -\frac{dS}{dt} - \frac{dR}{dt}\\ &= -S' - R'\\ &= \lambda IS - \gamma I \qquad (3)\\ &= (\lambda S - \gamma)I. \qquad (4)\end{aligned}$$

Following are some conclusions that can be made regarding this system of equations without solving them. Observe that by assumption $N(t)$ is constant so:

$$\begin{aligned} N(t) &= N(0) = N_0 \\ S(t) + I(t) + R(t) &= N_0 \\ S'(t) + I'(t) + R'(t) &= 0. \end{aligned}$$

Since S, I and λ are all positive, equation (1) tells us that $S'(t) < 0$ so,

$$\begin{aligned} S'(t) &< 0 \\ \therefore \quad S(t) &< S_0 \text{ for all } t > 0 \\ \text{So } \lambda S - \gamma &< \lambda S_0 - \gamma \\ (\lambda S - \gamma)I &< (\lambda S_0 - \gamma)I \\ I' &< (\lambda S_0 - \gamma)I. \end{aligned}$$

If $I' < 0$ then the epidemic fizzles out. So using this result with equation (3) from above we have that if $\lambda S_0 - \gamma < 0$ then the epidemic fizzles; this happens when

$$\begin{aligned} \lambda S_0 - \gamma &< 0 \\ \lambda S_0 &< \gamma \\ S_0 &< \frac{\gamma}{\lambda}. \end{aligned}$$

For a fixed disease we have no control over the constant γ. But we have some control over the "constant" λ: since we assume that $N(t)$ is the constant N_0 we have

$$\lambda = \frac{n\alpha}{N_0}$$

where, again, α is probability of an infection in an encounter between an infectious and a susceptible; it is a constant related to the disease. But we can decrease α by making it less likely that when a susceptible meets an infectious that there will be a new infection by using masks, and social distancing will decrease the constant n. Assuming we do this to obtain α' a smaller α. Then since we would need $I' < 0$, we want:

$$\begin{aligned} S_0 &< \frac{\gamma}{\lambda} \\ S_0 &< \frac{\gamma N_0}{n\alpha'} \\ n\alpha' S_0 &< \gamma N_0 \\ n &< \frac{\gamma N_0}{\alpha' S_0}. \end{aligned}$$

Assuming $S_0 \approx N_0$ so that $\frac{N_0}{S_0} \approx 1$ this gives us a way to combat an epidemic. Obviously the smaller the value of n, the smaller the average number of interactions between members of our population, the faster the epidemic will end. But we must at least have $n < \frac{\gamma}{\alpha'}$ in order to have any hope that the epidemic can be stopped. So the strategy of decreasing n by sequestering enough individuals well below the $n < \frac{\gamma}{\alpha'}$ threshold will eventually cause the epidemic to end.