Anotes01
Open Sets and Limit Points.

The object that we will study is the set of real numbers $\mathbb{R}$ or the number line and functions having domains that are subsets of the real numbers. [PS: I may generally refer to a number as a point on the number line.]

The "rules" for working with this object is the Axioms of the Real Numbers, it's posted on the class website. In addition, for the purposes of our study, all the rules about numbers with which you are familiar from your past mathematics courses. Typically these rules are called the Axioms of Arithmetic. [e.g. associative, commutative, distributive, identity elements, etc.] These axioms are those in the list of the axioms of the reals except for the completeness axiom. These axioms that are typically covered in MATH3100. In this course we will study the implications of this special axiom, the completeness axiom. It is necessary to prove all the rules that you learned in Calculus - so, since the purpose of this course is to develop the mathematical background for calculus from the axioms of the reals, you may not use any of the "rules" that you learned from calculus to prove the theorems, unless they have already been proven in class.

Each student should work on all the problems and try to prove all the theorems. Some of the problems will be impossible to do; in these cases you are expected to find counterexamples. [For the most part this is done deliberately; but occasionally a genuine error creeps in.]

Notation:

$\mathbb{R}$ denotes the set of all real numbers.

$\mathbb{Z}$ denotes the set of integers, $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Z}^+$ denotes the set of positive integers, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

$\mathbb{N}$ denotes the set of whole numbers, the non-negative integers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Problem 1.0 Suppose that $\epsilon > 0$. Show that there is a number x so that

$$0 < x < \epsilon.$$

[Note that Problem 1.0 implies that there is no smallest positive number.]

Problem 1.1. Suppose that a is a number and b is a number and it is true that if $\epsilon > 0$ then $|a - b| < \epsilon$. Show that $a = b$.

Notation: We use the following notation for segments and intervals of real numbers, if a is a number and b is a number then:

(a, b) denotes $\{x \in X | a < x < b\}$; this set is called a segment;
$[a, b]$ denotes $\{x \in X | a \leq x \leq b\}$; this set is called an interval;
$(a, b]$ denotes $\{x \in X | a < x \leq b\}$;
$[a, b)$ denotes $\{x \in X | a \leq x < b\}$.

Unless otherwise stated, sets will be assumed to be sets of real numbers.

If A is a set then A^c denotes the complement of A. (I.e. $A^c = \mathbb{R} - A$.)

Definition. The set of numbers M is said to be open if and only if for each point $x \in M$ there is a positive number ϵ [typically dependent upon x and M] so that the set $S = \{t| \ |t - x| < \epsilon\}$ lies in M (is a subset of M). [Helpful observation: $\{t| \ |t - x| < \epsilon\} = (x - \epsilon, x + \epsilon)$.]

Problem 1.2. Show that the set M is open if and only if for each $x \in M$ there is a segment S containing x so that

$$S \quad \subset \quad M.$$

Exercise 1.1. State what it means for a set not to be open and give an example of a set that is not open.

Exercise 1.2. Determine which of the following sets are open.

a. $[0, 1]$.
b. $(0, 1)$.
c. $\mathbb{R} - [0, 1]$.
d. $\mathbb{R} - (0, 1)$.
e. $\mathbb{R} - \{\frac{1}{n} | n \in \mathbb{N}\}$.
f. $\mathbb{R} - \{0\}$.
g. $\mathbb{Z}$ (the integers).
h. $\mathbb{R} - \mathbb{Z}$.
i. $\mathbb{R} - \{0\} - \{\frac{1}{n} | n \in \mathbb{N}\}$.
j. $\cup_{n=1}^{\infty} (\frac{1}{2n}, \frac{1}{2n-1})$.

Exercise 1.3. Show that the following sets are not open.

a. $\mathbb{N}$.

b. The rational numbers $\mathbb{Q}$.

c. $[0,1]$.

d. $\{x|x^2 - 5 \geq 0\}$.

Theorem 1.1. If A is open and B is open then $A \cup B$ and $A \cap B$ are open.

Definition. Suppose that M is a set, then $\text{int}(M)$ denotes the set to which x belongs if and only if there is a positive number ϵ so that $\{t\big||x-t| < \epsilon\} \subset M$. The set $\text{int}(M)$ is called the interior of M.

Exercise 1.4. Observe that a set U is open if and only $U = \text{int}(U)$. Then for each set in exercises 1.2 and 1.3 find the interior of the set. (And note by the previous observation that you've already done this for a good number of sets.)

Notation: If G is a collection of sets then $\cup G$ denotes the union of all the elements of G; $\cup G = \{x|$ there exists $g \in G$ so that $x \in g\}$. If G is a collection of sets then $\cap G$ denotes the common part of all the elements of G; $\cap G = \{x|x \in g$ for all $g \in G\}$.

Exercise 1.5. Determine if the following is true: If G is a collection of open sets then $\cup G$ and $\cap G$ are open.

Theorem 1.2. Formulate a theorem based on the previous exercise.

Examples.

1. Give an example of a set that contains no open set.
2. Give an example of a set so that neither it nor its complement contains an open set.

Definition. If M is a set of points then the point p is said to be a limit point of the set M if and only if every open set that contains p contains a point of M distinct from p.

Observation. The above definition for limit point is equivalent to the following:

A. The point p is a limit point of the set M if and only if every segment that contains p contains a point of M distinct from p.

B. The point p is a limit point of the set M if and only if for each $\epsilon > 0$ there is a point $x \in M$ so that $0 < |p - x| < \epsilon$.

[Note that the selection of x depends on the segment in A and on the choice of ϵ in B.]

Exercise 1.6. State what it means to say that the point p is not a limit point of the set M.

A helpful concept is that of a topological game. Such a "game" is typically played between two players. Depending on which player has a winning strategy, some important property of the space under consideration is revealed. In our case the space is the real numbers $\mathbb{R}$.

The Limit Point Game.

The game is for two players and begins with the players being given a point set M of real numbers and a point p (which need not belong to M); I may call the set M and point p as the "game board" for the particular game being played. I will designate the two players as player $\mathcal{O}$ and player $\mathcal{P}$.

Rules of the play:

Player $\mathcal{O}$ goes first by selecting a positive number ϵ_1.

Then for their turn, player $\mathcal{P}$ selects a point $x_1 \in M$ distinct from p so that $|p - x_1| < \epsilon_1$; if they cannot do this then $\mathcal{P}$ loses and the game ends. Otherwise the game continues with player $\mathcal{O}$ taking their next turn.

Player $\mathcal{O}$ selects another positive number ϵ_2. Then player $\mathcal{P}$ must select a point $x_2 \in M$ distinct from p so that $|p - x_2| < \epsilon_2$; again if they are unable to do this they lose and the game ends.

$$\vdots$$

The play continues in this fashion; if the game does not end then player $\mathcal{P}$ wins. (I.e. if player $\mathcal{P}$ does not lose at some turn after a finite number of steps then they win.)

Given the “infinite” nature of this game, in order for player $\mathcal{P}$ to claim to win, they must convince player $\mathcal{O}$ that they have a winning strategy that can always counter any of player $\mathcal{O}$'s moves.

Exercise 1.7. Prove that if $M = \{1 + \frac{1}{n} | n \in \mathbb{Z}\}$ then there is a point p so that player $\mathcal{P}$ has a winning strategy in the limit point game. (Hint: identify p first.)

Exercise 1.8. Prove that if $M = \{0, 1\}$ then player $\mathcal{O}$ has a winning strategy in the limit point game no matter what point p is selected.

Exercise 1.9. Consider the following Limit Point Game Boards. Determine for the following sets M which player has a winning strategy in the limit point game for the various choices of the starting point p. In particular for each game board, determine the set of points for which player $\mathcal{P}$ has a winning strategy.

a.) $M = \{2^{-n} | n \in \mathbb{Z}^+\}$.
b.) $M = \{2^{n} | n \in \mathbb{Z}^+\}$.
c.) $M = \{2^{-n} | n \in \mathbb{Z}\}$.
d.) $M = \{3 + \frac{1}{n} | n \in \mathbb{Z}, n \neq 0\}$.
e.) $M = \{\frac{n-1}{n+1} | n \in \mathbb{N}\}$.
f.) $M = \{\frac{2n+1}{3n+4} | n \in \mathbb{N}\}$.
g.) $M = \{\frac{n+1}{n^2+1} | n \in \mathbb{N}\}$.
h.) $M = \{\frac{n}{m} | n, m \in \mathbb{Z}^+\}$.
i.) $M = \{\frac{n}{m} | n, m \in \mathbb{Z}^+, n < m\}$.
j.) $M = \{\frac{n}{m} | n, m \in \mathbb{Z}^+, n > m\}$.
k.) $M = \{\frac{n}{m} | n, m \in \mathbb{Z}^+, \frac{n^2}{m^2} > 2\}$.
l.) $M = \{\sin(n\pi) | n \in \mathbb{N}\}$.
m.) $M = \{\sin(n) | n \in \mathbb{N}\}$.
n.) $M = \{\arctan(n\pi) | n \in \mathbb{N}\}$.
o.) $M = \{\arctan(n) | n \in \mathbb{N}\}$.
p.) $M = \{x | x \in [-1, 1]\}$.
q.) $M = \{x | x < 2\}$.

Theorem 1.3. The point p is a limit point of the set M if and only if every open set containing p contains a point of M distinct from p.

[Hint: note that the point x is dependent upon the choice of open set.]

More examples, give examples of:

3. A set that has a limit point.
4. A set that has no limit points.
5. A set every point of which is a limit point of it.
6. A set every point of which is a limit point of it and whose complement also has this property.
7. A set so that it and its complement have no common limit points.
8. A set so that every point of it is a limit point of the set and is also a limit point of its complement.

Definition. A set is said to be closed if it contains all of its limit points.

Examples, continued, give examples of:

9. A set that is closed.
10. A set that is not closed.
11. A set so that neither it nor its complement is closed.

Notation. If M is a set then $\overline{M}$ denotes the set to which x belongs if and only if x is an element of M or x is a limit point of M.

Lemma to Theorem 1.4. If M is a point set then $\overline{M}$ is closed.

Theorem 1.4. If M is a set of points, then:

$$\overline{M} = \overline{\overline{M}}.$$

Theorem 1.5. If the set M has a limit point then M is an infinite set.

Examples, give examples of:

12. A set that has exactly one limit point.
13. A set that has exactly two limit points.
14. A set whose set of limit points has exactly one limit point.
15. A set every point of which is a limit point but which contains no segment.
16. A closed set every point of which is a limit point but which contains no segment.

Theorem 1.6. If A is a set then A is open if and only if A^c is closed.

Background Theorem. If $a < b$ then:

1. there is a rational number between a and b, and
2. there is an irrational number between a and b.

Theorem 1.7. If A and B are sets and p is a limit point of the set $A \cup B$, then either p is a limit point of A or p is a limit point of B.

Open Sets, Limit Points and Boundary Points.

Definition. If M is a pointset, then the point p is said to be a boundary point of M if and only if every segment containing p also contains a point in M and a point not in M.

Observation. The above definition for boundary point is equivalent to the following: The point p is a boundary point of the set M if and only if for each $\epsilon > 0$ the segment $(p - \epsilon, p + \epsilon)$ contains a point of M and a point not in M.

Following are theorems about the relationship of open sets, boundary points, limit points and closed sets.

Theorem 1.8. If M is a pointset, $p \in M$ and p is a limit point of $\mathbb{R} - M$ then M is not open.

Theorem 1.9. An open set contains none of its boundary points.

Theorem 1.10. If the point p is a boundary point of M and $p \in M$, then p is a limit point of $\mathbb{R} - M$.

Theorem 1.11. If the point p is a boundary point of M and $p \notin M$, then p is a limit point of M.

Theorem 1.12. The point p is a boundary point of the set M if and only if p is a boundary point of $\mathbb{R} - M$.