Anotes02
Upper Bounds and Least Upper Bounds.

Definition. If M is a pointset, the the number B is said to be an upper bound for M if and only if $x \leq B$ for all $x \in M$.

Definition. If M is a pointset, the number L is said to be a least upper bound of M if and only if L is an upper bound for M and if $L' < L$ then L' is not an upper bound for M. The least upper bound of the set M is denoted by $\sup(M)$.

[Note that lower bounds and greatest lower bounds are similarly defined; the greatest lower bound of the set M is denoted by $\inf(M)$.]

Exercise 2.1.

a.) Find an example of a set M and a least upper bound L for M so that $L \in M$.

b.) Find an example of a set M and a least upper bound L for M so that $L \notin M$.

Theorem 2.1. If M is a set and each of B_1 and B_2 is a least upper bound for M then $B_1 = B_2$.

Exercise 2.2. Define what a lower bound and a greatest lower bound for a set would mean.

Based on this exercise use the least upper bound axiom to prove:

Theorem. If M is a set and M has a lower bound, then M has a greatest lower bound.

Exercise 2.3. For each of the sets M of the Limit Point gameboards of the set of notes Anotes01, determine the least upper bound of M.

Theorem 2.2. If M is a set and L is the least upper bound for M and $L \notin M$, then L is a limit point of M.

Theorem 2.3. If M is a point set and p is a limit point of M so that p is greater than every point of M, then p is the least upper bound of M.

Theorem 2.4. Suppose that M is a point set and L is the least upper bound of M. Let $K = \{B|B \text{ is an upper bound for } M\}$ then L is the greatest lower bound for K.

Theorem 2.5. Suppose that M_1 and M_2 are sets with least upper bounds L_1 and L_2. Let $M = \{x+y|x \in M_1, y \in M_2\}$; then $L_1 + L_2$ is the least upper bound for M.

Exercise 2.4. Suppose that M_1 and M_2 are sets with least upper bounds L_1 and L_2. Let $M = \{xy|x \in M_1, y \in M_2\}$; then what can be said about the least upper bound for M.

Theorem 2.6. Suppose that M_1 and M_2 are sets with least upper bounds L_1 and L_2. Then either L_1 or L_2 is the least upper bound of $M_1 \cup M_2$.

Exercise 2.5. Compose and prove a theorem like theorem 2.6 for $M_1 \cap M_2$.

Definition. For a set M we let M' denote the set of limit points of M. Note that points of M' may or may not be points of M.

Exercise 2.6. For the following sets M, determine M', $\sup(M)$ and $\inf(M)$; in each determine if $\sup(M)$ is a point of the set or limit point of the set and similarly for $\inf(M)$. [Hint: sketch the sets, for v-vii sketching the function will help you sketch the set.]

i. $M = \{\frac{1}{n}|n \in \mathbb{N}\} \cup \{\frac{3n-1}{n+1}|n \in \mathbb{N}\}$,

ii. $M = ([0,1] - \{\frac{1}{2}\}) \cup ([3,4] \cap \mathbb{Q})$.

iii. $M = \{x \in R|(x^2-4)^2 < 3\}$,

iv. $M = ([0,1] - Q) \cup (3,4)$,

v. $M = \{e^{-x^2}|x \in \mathbb{R}\}$,

vi. $M = \{\frac{x^2+3}{x^2+1}|x \in \mathbb{R}\}$,

vii. $M = \{\frac{2x-1}{3x+1}|x \in \mathbb{R}, x > 1\}$.