Anotes03
Sequential Limit Points.

Definition: A sequence is a function from the positive integers to the real numbers.

Notation: If x is a sequence then x_n denotes the value of the sequence associated with the integer n and the sequence itself will be denoted by $\{x_n\}_{n=1}^{\infty}$ or $x_1, x_2, x_3, \ldots$

Definition: The point P is said to be the sequential limit of the sequence $\{x_n\}_{n=1}^{\infty} = x_1, x_2, x_3, \ldots$ if and only if it is true that if S is an open set containing P then there is an integer N so that $x_n \in S$ for all $n > N$.

Observe that the choice of N depends on the open set S.

Definition: If A is a sequence $\{x_n\}_{n=1}^{\infty}$ then the sequence is said to converge if it has a sequential limit point, otherwise it is said to diverge.

Theorem 3.1. A sequence has at most one sequential limit point.

Theorem 3.15. If P is the sequential limit point of $\{x_n\}_{n=1}^{\infty}$, then 0 is the sequential limit of the sequence $\{x_n - x_{n+1}\}_{i=1}^{\infty}$.

Exercise 3.15. Is the converse of Theorem 3.15 true (or something close to the converse.)

Definition. The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be an increasing sequence if and only if for each positive integer i we have $x_i < x_{i+1}$.

Theorem 3.2. Suppose that P is the sequential limit point of $\{x_n\}_{n=1}^{\infty}$ and $N_1, N_2, N_3, \ldots$ is an increasing sequence of positive integers. Then P is also the sequential limit point of $\{x_{N_i}\}_{i=1}^{\infty}$.

Lemma to Theorem 3.3. The point p is the sequential limit of the sequence $\{x_n\}_{n=1}^{\infty}$ if an only if the following condition holds [the $\epsilon - N$ condition.]:

If $\epsilon > 0$ then there exists an integer N so that if $n > N$ then:

$$|p - x_n| < \epsilon.$$

[Many textbooks use this as the definition of sequential limit point. Note that N is dependent upon ϵ and will typically be different for different values of ϵ; I may emphasize this by denoting the integer that "goes" with $\epsilon > 0$ by N_ϵ.]

Theorem 3.3. Suppose that A is the sequential limit point of the sequence $\{a_n\}_{n=1}^{\infty}$ and B is the sequential limit point of the sequence $\{b_n\}_{n=1}^{\infty}$. Then $A + B$ is the sequential limit point of the sequence $\{a_n + b_n\}_{n=1}^{\infty}$.

Theorem 3.4. Suppose that A is the sequential limit point of the sequence $\{a_n\}_{n=1}^{\infty}$ and c is a number. Then cA is the sequential limit point of the sequence $\{c \cdot a_n\}_{n=1}^{\infty}$.

Theorem 3.5. Suppose that A is the sequential limit point of the sequence $\{a_n\}_{n=1}^{\infty}$ and B is the sequential limit point of the sequence $\{b_n\}_{i=1}^{\infty}$. Then $A \cdot B$ is the sequential limit point of the sequence $\{a_n \cdot b_n\}_{n=1}^{\infty}$.

Lemma to Theorem 3.5. Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges and $M = \{a_n |\ n \in Z^+\}$ is the set of terms of the sequence. Then M is bounded.

Theorem 3.6. Suppose that M is a set and p is a limit point of M. Then there exists a sequence of distinct points of M so that p is the sequential limit point of that sequence.

Exercise 3.2. Determine which of the following sequences converge; if it converges, using the results of lemma 3.3 calculate N_ϵ and then prove that it converges; if it diverges, prove that it diverges.

a.) $\{\frac{1}{n}\}_{n=1}^{\infty}$.
b.) $\{2^{-n}\}_{n=1}^{\infty}$.
c.) $\{n^2\}_{n=1}^{\infty}$.
d.) $\{3 + \frac{1}{n}\}_{n=1}^{\infty}$.
e.) $\{\frac{n-1}{n+1}\}_{n=1}^{\infty}$.
f.) $\{(-1)^n \frac{n-1}{n+1}\}_{n=1}^{\infty}$.
g.) $\{\frac{2n+1}{3n+4}\}_{n=1}^{\infty}$.
h.) $\{\frac{(-1)^n}{n+1}\}_{n=1}^{\infty}$.
i.) $\{\sum_{i=1}^{n} 2^{-i}\}_{n=1}^{\infty}$.

j.) $\{\sum_{i=1}^{n}(-2)^{-i}\}_{n=1}^{\infty}$.
k.) $\{\sum_{i=1}^{n}\frac{1}{i}\}_{n=1}^{\infty}$.
l.) $\{\sum_{i=1}^{n}(-1)^{i}\frac{1}{i}\}_{n=1}^{\infty}$.