Anotes07
Uniform continuity; uniform convergence.

Definition. Suppose that $M \subset R$. Then the function $f : M \to \mathbb{R}$ is said to be uniformly continuous over its domain means that if $\epsilon > 0$ is a positive number then there is a positive number δ so that $|f(x) - f(y)| < \epsilon$ for all x, y so that $|x - y| < \delta$.

Exercise 7.1. Determine if the following functions are uniformly continuous over the indicated sets M :

$$
\begin{array}{lll}
a.) & f(x) = \frac{3x+1}{5} & M = \mathbb{R} \\
b.) & f(x) = x^2 & M = [0, 2] \\
c.) & f(x) = x^2 & M = \mathbb{R} \\
d.) & f(x) = \frac{1}{x} & M = [1, 3] \\
e.) & f(x) = \frac{1}{x} & M = (0, 3) \\
f.) & f(x) = \frac{1}{x^2+1} & M = \mathbb{R}
\end{array}
$$

Definition. Suppose that $f_1, f_2, f_3, ...$ is a sequence of functions with common domain D. Then the sequence of functions is said to converge pointwise on the domain D to the function f if and only if for each $p \in D$, $f(p)$ is the sequential limit of the sequence $\{f_i(p)\}_{i=1}^{\infty}$.

Observe that this means that if $x \in D$ and $\epsilon > 0$ then there exists an integer N so that if $n > N$ then $|f(x) - f_n(x)| < \epsilon$. Observe also, as indicated in class, that the value of N may depend on x.

Definition. Suppose that $f_1, f_2, f_3, ...$ is a sequence of functions with common domain D. Then the sequence of functions is said to converge uniformly on the domain D to the function f if and only if for each $\epsilon > 0$, there exists an integer N so that if $n > N$ then $|f(x) - f_n(x)| < \epsilon$ for all $x \in D$.

Exercise 7.2. Show that pointwise convergence is different from uniform convergence by finding a sequence of functions that does one but not the other.

Exercise 7.3. Show that if the sequence $f_1, f_2, f_3, ...$ converges uniformly to f then it converges pointwise to f.

Exercise 7.4. Show that the sequence of functions $f_n = \frac{x+1}{n}$ converges uniformly to the function $f(x) = 0$ on the interval $[0, 2]$.

Theorem 7.1. If $f_1, f_2, f_3, ...$ is a sequence of continuous functions with common domain the interval I that converges uniformly, then this sequence of functions converges to a continuous function.

Exercise 7.5.

a. Find a sequence of discontinuous functions that converges pointwise to a continuous function.

b. Find a sequence of discontinuous functions that converges uniformly to a continuous function.

Theorem 7.2. Suppose that f is continuous on the interval $[a, b]$. Then f is bounded on $[a, b]$. (By which is meant that there is a number M so that $|f(x)| < M$ for all $x \in [a, b]$.)

Theorem 7.3. Suppose that f is continuous on the interval $[a, b]$. Then f is uniformly continuous on $[a.b]$.

Theorem 7.4. Suppose that f is continuous on the compact set M. Then f is bounded on M.

Theorem 7.5. Suppose that f is continuous on the compact set M. Then f is uniformly continuous on M.