Anotes08
Countable and Uncountable Sets.

Definition. Suppose that M is a set. Then M is said to be countable if and only if M is a finite set or there exists a 1-1 onto function from the natural numbers onto M. [Equivalently: there exists a function from a subset of $\mathbb{N}$ onto M.]

Definition. The set M is said to be uncountable if it is not countable.

Exercise 8.1. Show that the set $\mathbb{Z}$ of all the integers is countable.

Exercise 8.2. Show that the set of all integral multiples of $\frac{1}{2}$ is countable.

Theorem 8.1. If M is a countable set and $H \subset M$ then H is countable.

Theorem 8.2. If each of H and K is countable then the set $H \cup K$ is countable.

Theorem 8.3. If each of H and K is countable then the set $H \times K$ is countable.

Corollary 8.3. The set of rational numbers is countable.

Theorem 8.4. If for each integer n, the set H_n is countable, then the set $\cup_{i=1}^{\infty} H_n$ is countable.

Theorem 8.5. If G is a collection of disjoint segments (i.e. each two elements of G do not intersect) then G is countable

Definition. The set M is said to be dense in the reals $\mathbb{R}$ if and only if every segment contains a point of M.

Exercise 8.3. In each case find a function with the indicated property.
a. $f : [0, 1] \to R$ so that the set of points of discontinuity is countably infinite.
b. $f : [0, 1] \to R$ so that the set of points of discontinuity is dense in $[0, 1]$ and countably infinite.

Definition: The function f is said to be increasing on its domain D if whenever $x < y$ we have $f(x) < f(y)$.
c. & d. Repeat a. & b. but require that f be increasing.

Definition. The set M is said to be nowhere dense if and only if for each segment S there exists a segment $U \subset S$ so that $U \cap M = \emptyset$.

Theorem 8.6. If M is nowhere dense and $H \subset M$ then H is nowhere dense.

Theorem 8.7. If M is nowhere dense then so is $\overline{M}$.

Theorem 8.8. A finite set is nowhere dense.

Theorem 8.9. If each of H and K is nowhere dense then so is $H \cup K$.

Theorem 8.10. If I is a closed interval then I is not the union of countably many nowhere dense sets.

Hint. There is a proof that uses the fact that every monotonic collection of intervals has a common point. There is a proof that uses absolute values and the fact that every Cauchy sequence converges.

Corollary 8.10.1. The set of real numbers is uncountable.

Corollary 8.10.2. If $\{U_i\}_{i=1}^{\infty}$ is a countable collection of dense open sets in the reals, then every open set contains an element of $\cap_{i=1}^{\infty} U_i$.

Theorem 8.11. If I is a closed interval then I is not the union of countably many disjoint intervals.

Exercise 8.4 Suppose that M is an uncountable subset of the reals. Show:

a. M has a limit point.

b. M contains one of its limit points.

Exercise 8.5. Find a nowhere dense set such that every point of the set is a limit point of it.