Anotes09
Infinite Series.

Notation. If the sequence $\{x_n\}_{n=1}^{\infty}$ has a sequential limit p, then this limit is denoted by:

$$p = \lim_{n\to\infty} x_n.$$

Definition [Of the infinite sum notation]. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers. If the sequence

$$\sum_{i=1}^{n} x_i$$

converges to a sequential limit L, then

$$L = \sum_{i=1}^{\infty} x_i.$$

The terms $\sum_{i=1}^{n} x_i$ are called partial sums of the series.
If the sequence does not converge, then it is said to diverge.

Exercise 9.1. Use the definition of the sum of an infinite series to do the following exercises.

a. Show that $\sum_{n=1}^{\infty} 1$ diverges.

b. Show that $\sum_{n=0}^{\infty} (-1)^n$ diverges.

c. Determine the value of $\sum_{n=1}^{\infty} \frac{1}{2^n}$

[General hint: The Cauchy convergence criterion may be helpful in the proofs of these theorems.]

Theorem 9.1. If the series $\sum_{i=1}^{\infty} x_i$ converges then $\lim_{n\to\infty} x_n = 0$.

Exercise 9.2. Give an example to show that the converse is not true.

Theorem 9.2. [Geometric Series.] If $|r| < 1$ then the series $\sum_{n=0}^{\infty} r^n$ converges.

Theorem 9.3. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Theorem 9.4. If $\sum_{n=1}^{\infty} a_n$ converges and c is a real number then $\sum_{n=1}^{\infty} c \cdot a_n$ converges and

$$\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n.$$

Exercise 9.3. Is the following true: if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges then

$$\sum_{n=1}^{\infty} a_n \cdot b_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n.$$

Theorem 9.5. [Comparison Test Part I.] If for each n, $0 \leq b_n \leq a_n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

Exercise 9.4. Suppose that for each n, $a_n > 0$ and that $\sum_{n=1}^{\infty} a_n$ diverges. Show that if $B > 0$ then there exists an integer N so that

$$\sum_{n=1}^{N} a_n > B.$$

Note that this is often used as the definition of what it means for a limit of the sequence to go to ∞.

Theorem 9.6. [Comparison Test Part II.] If for each n, $0 \leq b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 9.7. [Limit Comparison Test.] Suppose that for each n, $0 \leq b_n$, $0 \leq a_n$ and that the sequence $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ converges to a positive number L. Then:

a. if $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} b_n$;
b. if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

Theorem 9.8. [Alternating Series Test.] Suppose that for each n, $a_n > 0$, $a_n > a_{n+1}$ and that $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Theorem 9.9. If $\sum_{n=1}^{\infty} |a_n|$ converges then so does $\sum_{n=1}^{\infty} a_n$.

Exercise 9.5. Show that the converse of theorem 9.9 is not true.

Definition. If $\sum_{n=1}^{\infty} a_n$ is a series so that $\sum_{n=1}^{\infty} |a_n|$ converges then the original series is said to be *absolutely convergent.*

Theorem. 9.10. [Ratio Test.] Suppose that for each n, $a_n > 0$ and $r < 1$ is a number so that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < r.$$

Then $\sum_{n=1}^{\infty} a_n$ converges.