Math 5200 Review of the rational numbers.
Constructing the rationals from the integers.

Since we will be using the properties of the integers for our construction, you may only use the axioms of the integers.

As a 5200 student, should be able to prove all the theorems stated below. For the project as given to my MATH 3100 Class: Prove theorems 0.2, 0,3, 0.5 and 0.6; also solve exercise 0.2; for theorem 0.4, just prove that $\mathbb{Q}$ satisfies axioms B5 and C1.

[Hint: Although it is not necessary, it may be helpful in formulating proofs, for you to remember about how the elements of $\mathbb{Q}$ are related to the rational numbers as you learned about them in grammar school: the equivalence class $[(a, b)]$ corresponds to the fraction $\frac{a}{b}$.]

As usual $\mathbb{Z}$ is the set of integers. We will be working with the set $\mathbb{Z}\times(\mathbb{Z}-\{0\})$; this is the set of all integer pairs whose second element is non-zero. We define an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Definition. Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define $\sim$ on S as follows:

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

Theorem 0.1. The relation $\sim$ is an equivalence relation on S.

Proof Outline. We need to show that the relation $\sim$ satisfies the conditions for being an equivalence relation.

1.) Reflexive: This means that $(a, b) \sim (a, b)$ and is easy.

2.) Symmetric: This one is easy too; write down what $(a, b) \sim (c, d)$ means and what $(c, d) \sim (a, b)$ means and compare.

3.) Transitive: This is the tricky one. Given $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ we want to conclude that $(a, b) \sim (e, f)$. Let's "translate" using the definition:

Given $ad = bc$ and $cf = de$ we want to conclude that $af = be$. Now if we were in the reals it would be easy; solve the second equation for d and substitute into the first and do the algebra. But this involves division and we don't have division for the integers. Recall that the cancellation rule can be sometimes used in place of division and use that to complete the proof. $\square$

Notation. The symbol $\mathbb{Q}$ will be used to denote the collection of equivalence classes: $\{[(a,b)]\ |a,b \in \mathbb{Z}, b \neq 0\}$.

Exercise 0.1. Determine list four or five elements of each of the following equivalence class:

$$\begin{array}{ll} a.) & [(0,1)] \\ b.) & [(2,1)] \\ c.) & [(1,2)] \\ d.) & [(2,3)] \\ e.) & [(3,2)] \\ f.) & [(1,1)] \\ g.) & [(-1,3)] \\ h.) & [(1,-3)]. \end{array}$$

Definition. The operations $+$ and $\cdot$ are defined as follows:

$$\begin{aligned} [(a,b)] + [(c,d)] &= \Big[(ad+bc, bd)\Big] \\ [(a,b)] \cdot [(c,d)] &= \Big[(ac, bd)\Big]. \end{aligned}$$

[Comment: The $+$ on the right operates on elements of S whereas the one on the left operates on integers; this shouldn't cause any confusion since the plus signs will always be between the same type of elements. It's sort of like, when you were in 2660 you saw the instructor use the symbol $\cdot$ to indicate multiplication of matrices even though they also used it to indicate multiplication of numbers. If I am concerned that there may be the possibility of confusion, I will denote the operations on $\mathbb{Q}$ as $+_\sim$ and $\cdot_\sim$.]

Theorem 0.2. The operations $+_\sim$ and $\cdot_\sim$ are well-defined.

Exercise 0.2. The operation $\oplus$ defined as follows is not well-defined.

$$[(a,b)] \oplus [(c,d)] = [(a+c, b+d)].$$

Theorem 0.3. There exist additive and multiplicative identities associated with the operations of addition $+$ and multiplication $\cdot$ on $\mathbb{Q}$.

Theorem 0.4. The A, B and C axioms of the integers are satisfied by $\mathbb{Q}$. [Most of these will be straight forward, the existence of the multiplicative and additive identities is claimed by theorem 0.3; given $[(a,b)]$ you will have to determine its additive inverse. This is really 11 theorems in 1, one for each axiom.]

Theorem 0.5 A. If $[(a,b)] \in \mathbb{Q}$ then $[(a,b)]$ has an additive inverse.

Theorem 0.5 B. If $[(a,b)] \neq [(0,1)]$ then $[(a,b)]$ has a multiplicative inverse.

Theorem 0.6. Let $\varphi : (\mathbb{Z}, +.) \to (\mathbb{Q}, +_\sim, \cdot_\sim)$ be defined as follows:

$$\varphi(n) = [n, 1].$$

Then φ is a homomorphism with respect to both addition $+$ and $+_\sim$ and multiplication $\cdot$ and $\cdot_\sim$ operators.