MATH 5200 Take Home Project 02 Key.

The project is due Monday Feb. 21 by the beginning of class. You are allowed to work together, but if you do you must indicate which person (or persons) you worked with and the critical contributions of that person (or those persons.)

Provide solutions to all the problems, I will base the grade on the best 5 out of 6.

As usual, email to me a pdf copy of your work with your last name as the first part of the file name.

1. Prove that the compliment of an open set is a closed set.

Proof. Let M be an open set. The problem is to prove that the complement of M is closed. To that end let $p \in M$. Then since M is open there exists a number $\epsilon > 0$ so that $p \in (p-\epsilon, p+\epsilon) \subset M$. But then $(p-\epsilon, p+\epsilon)$ contains no points of the complement of M so p is not a limit point of this complement. So, since M contains no limit points of its complement, the complement must contain all of its limit point and so is closed. □

2. Consider the sequence

$$S = \left\{ \frac{2\sqrt{n}-5}{\sqrt{n}+2} \,\middle|\, n \in \mathbb{N} \right\}.$$

Find the sequential limit point of S and prove that the point you found is the sequential limit point of S.

Solution. [Note: scratch is below so that you can see how I arrived at the number N_ϵ.]

We claim that the sequential limit of the sequence is 2. To prove this we let $\epsilon > 0$ and select N_ϵ to be an integer greater than $\frac{81}{\epsilon^2}$. Then if $n > N_\epsilon$ we have:

$$
\begin{aligned}
\frac{81}{\epsilon^2} &< n \\
\frac{9}{\epsilon} &< \sqrt{n} \\
\frac{9}{\epsilon} &< \sqrt{n}+2 \\
9 &< \epsilon(\sqrt{n}+2) \\
9 &< \epsilon\sqrt{n}+2\epsilon \\
-\epsilon\sqrt{n}-2\epsilon+4 &< -5 \\
2\sqrt{n}-\epsilon\sqrt{n}-2\epsilon+4 &< 2\sqrt{n}-5 \\
(2-\epsilon)(\sqrt{n}+2) &< 2\sqrt{n}-5 \\
2-\epsilon &< \frac{2\sqrt{n}-5}{\sqrt{n}+2}.
\end{aligned}
$$

Thus the n^{th} term of the sequence is greater than $2-\epsilon$; I also need to prove that it's less than $2+\epsilon$. I claim that it is in fact less than 2 which places it in the segment. To prove that, suppose not and that for some $n > N_\epsilon$ we have

$$
\begin{aligned}
2 &< \frac{2\sqrt{n}-5}{\sqrt{n}+2} \\
2\sqrt{n}+4 &< 2\sqrt{n}-5 \\
4 &< -5.
\end{aligned}
$$

Since the last equation is impossible it follows that for $n > N_\epsilon$ we have:

$$2-\epsilon < \frac{2\sqrt{n}-5}{\sqrt{n}+2} < 2+\epsilon.$$

So 2 is the sequential limit of the sequence. □

***************************Scratch work**********************

Long division gives us:

$$\frac{2\sqrt{n}-5}{\sqrt{n}+2} = 2 - \frac{9}{\sqrt{n}+2}$$

this tells me that the sequence is increasing to the limit 2. So we can either set

$$\frac{9}{\sqrt{n}+2} < \epsilon$$

or

$$2 - \epsilon < \frac{2\sqrt{n}-5}{\sqrt{n}+2};$$

in either case you'll get the same inequality. I'll proceed with the second method which most of you have used:

$$\begin{aligned} 2 - \epsilon &< \frac{2\sqrt{n}-5}{\sqrt{n}+2} \\ 2\sqrt{n} - \epsilon\sqrt{n} + 4 - 2\epsilon &< 2\sqrt{n} - 5 \\ 9 &< \epsilon(\sqrt{n}+2) \\ \frac{9}{\epsilon} &< \sqrt{n}+2 \\ \left(\frac{9}{\epsilon} - 2\right)^2 &< n. \end{aligned}$$

***********************End Scratch work**********************

3. Consider the following sequence

$$S = (-1)^n \left\{\frac{1}{5} + \frac{7}{n}\right\}_{n=1}^{\infty}.$$

a.) Prove that $\frac{1}{5}$ is a limit point of the set of terms of the sequence.

Proof. Let $\epsilon > 0$. Let n be an even integer greater than $\frac{7}{\epsilon}$. Then

$$\frac{1}{5} - \epsilon < \frac{1}{5} < \frac{1}{5} + \frac{7}{n} < \frac{1}{5} + \epsilon.$$

Since for the even number n, the n^{th} term of the sequence is $\frac{1}{5} + \frac{7}{n}$, which is an element distinct from $\frac{1}{5}$ in the segment $(\frac{1}{5} - \epsilon, \frac{1}{5} + \epsilon)$. □

b.) Prove that the number $\frac{1}{5}$ is not the sequential limit of this sequence.

c.) Show that the sequence does not have a sequential limit.
[Note that doing (c) implies the solution to (b) - I set it up this way because working through (b) should help you do (c).]

Proof. Let

$$x_n = (-1)^n\Big(\frac{1}{5} + \frac{7}{n}\Big).$$

Suppose that the sequence has sequential limit p. The for $\epsilon = \frac{1}{5}$ there would exist an integer N so that if $n > N$ then $|p - x_n| < \epsilon = \frac{1}{5}$. Let n be an even integer with $n > N$. Then

$$\begin{aligned}
|p - x_n| + |p - x_{n+1}| &< \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \\
|(x_n - p) + (p - x_{n+1})| &\leq |x_n - p| + |p - x_{n+1}| < \frac{2}{5} \\
|(x_n - x_{n+1})| &\leq |x_n - p| + |p - x_{n+1}| < \frac{2}{5} \\
\Big|\Big(\frac{1}{5} + \frac{7}{n}\Big) - -\Big(\frac{1}{5} + \frac{7}{n+1}\Big)\Big| &< \frac{2}{5} \\
\Big|\Big(\frac{1}{5} + \frac{7}{n}\Big) + \Big(\frac{1}{5} + \frac{7}{n+1}\Big)\Big| &< \frac{2}{5} \\
\frac{2}{5} + \frac{7(n+1) + 7n}{n(n+1)} &< \frac{2}{5} \\
\frac{2}{5} &< \frac{2}{5}.
\end{aligned}$$

The forth line up from the bottom follows from the fact that n was chosen to be even and so $n+1$ is odd; the second line up from the bottom follows from the fact that all the numbers inside the absolute value symbol are positive; and the last line follow from the fact that n is positive and fraction $\frac{7(n+1)+7n}{n(n+1)}$ is positive. This yields the contradiction of the last line, so the assumption that the sequence has a sequential limit is false. □

4.) Prove, from the definition of sequential limit point, that if a is the sequential limit point of the sequence $\{a_n\}_{n=1}^{\infty}$ then a^2 is the sequential limit point of the sequence $\{a_n^2\}_{n=1}^{\infty}$.

[Hint 1: $a^2 - a_n^2 = (a - a_n)(a + a_n)$.]

[Hint 2: Prove that the set $M = \{a + a_n | n \in \mathbb{Z}^+\}$ is bounded and look at the proof of theorem 3.4.]

Proof. We know from hint 2 (proven in class) that the set $M = \{|a + a_n| \big| n \in \mathbb{Z}^+\}$ is bounded. Let B be an upper bound for M.

Let $\epsilon > 0$. Then, since $\{a_n\}_{n=1}^{\infty}$ has sequential limit a, there is an integer N so that if $n > N$ then

$$|a - a_n| \;<\; \frac{\epsilon}{B}.$$

Therefore if $n > N$ then

$$\begin{aligned} |a^2 - a_n^2| &= |(a - a_n)(a + a_n)| \\ &= |a - a_n|\;|a + a_n| \\ &< \frac{\epsilon}{B} \cdot B \\ &< \epsilon. \end{aligned}$$

So the sequence $\{a_n^2\}_{n=1}^{\infty}$ has sequential limit a^2. □

5.) Suppose that $\{M_i\}_{i \in I}$ and that for each $i \in I$, L_i is the least upper bound of M_i. Prove that if L is the least upper bound of $\cup_{I \in I} M_i$, then L is the least upper bound of the set $\{L_i | i \in I\}$.

Solution. Suppose that L is the least upper bound of $\cup_{I \in I} M_i$. Then L must be an upper bound for M_i; so $L_i \leq L$ for all $i \in I$. So L is an upper bound of the set $\{L_i | i \in I\}$. Suppose that $L' < L$. Then since L is the least upper bound of $\cup_{I \in I} M_i$ it follows that there is a point x of $\cup_{I \in I} M_i$ greater than L'. So x belongs to some M_i for some $i \in I$. So $x \leq L_i$. So L_i is an element of the set $\{L_i | i \in I\}$ which is greater than L'. Therefore L' is not an upper bound for this set and so L is the least upper bound of the set. □

6.) Prove theorem 4.2.
[Hint: the proof requires the least upper bound axiom.]