MATH 5200
Test, April 15, 2022.

Show all your scratch work. Indicate your scratch work; no credit will be taken off for errors in the scratch work. If you do both parts of #4 I will base your grade on the better of the two; may do both proofs for possible extra credit. The test is closed book and closed notes, you may not use any electronic equipment during the exam. Make sure you are socially distanced from your nearest neighbor.

Problem 1. Prove that the function $f(x) = \sqrt{3x+2}$ is uniformly continuous over the interval $[0, 5]$.

Proof. Let $\epsilon > 0$. Then select $0 < \delta_\epsilon < \epsilon\frac{2\sqrt{2}}{3}$. Then if $|x - p| < \delta_\epsilon$:

$$\begin{aligned}
|\sqrt{3x+2} - \sqrt{3p+2}| &= \left|\frac{3x+2-3p-2}{\sqrt{3x+2}+\sqrt{3p+2}}\right| \\
&= \left|\frac{3(x-p)}{\sqrt{3x+2}+\sqrt{3p+2}}\right| \\
&= |x-p|\left|\frac{3}{\sqrt{3x+2}+\sqrt{3p+2}}\right| \\
&< \delta_\epsilon\left|\frac{3}{\sqrt{3x+2}+\sqrt{3p+2}}\right| \\
&< \delta_\epsilon\frac{3}{2\sqrt{2}} \qquad (1) \\
&< \epsilon\frac{2\sqrt{2}}{3}\frac{3}{2\sqrt{2}} = \epsilon.
\end{aligned}$$

Step (1) follows from the fact that the smallest values for x or p in the interval is 0 and this makes the expression as large as possible. □

Problem 2. Show that the following sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly over the interval $[0, 3]$ to a function:

$$f_n(x) = 5x^2 + \frac{x^2}{n}.$$

Proof. We claim that this sequence of functions approaches the function $f(x) = 5x^2$. Let $\epsilon > 0$. Then select N to be an integer so that $N > \frac{9}{\epsilon}$. Then if $n > N$ we have $\frac{1}{n} < \frac{\epsilon}{9}$; so:

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|5x^2 + \frac{x^2}{n} - 5x^2\right| \\
&= \frac{x^2}{n} \\
&\leq \frac{9}{n} && (2) \\
&< 9\frac{\epsilon}{9} = \epsilon.
\end{aligned}$$

Where step (2) follows from the fact that $x = 3$ makes the expression the biggest possible over the interval $[0, 3]$.

$\square$

Problem 3. Show that the following function is uniformly continuous over $[1, \infty)$.

$$f(x) = \frac{5}{3x^2}.$$

Proof. Let $\epsilon > 0$. Then select $0 < \delta_\epsilon < \epsilon\frac{3}{10}$. Then if $|x - p| < \delta_\epsilon$:

$$\begin{aligned}
\left|\frac{5}{3x^2} - \frac{5}{3p^2}\right| &= 5\left|\frac{3p^2 - 3x^2}{3x^2 \cdot 3p^2}\right| \\
&= 5\left|\frac{(p-x)(p+x)}{x^2 \cdot 3p^2}\right| \\
&= |p - x| 5\left(\frac{1}{x^2 \cdot 3p} + \frac{1}{x \cdot 3p^2}\right) \\
&\leq |p - x| 5\left(\frac{1}{3} + \frac{1}{3}\right) && (3) \\
&\leq |p - x|\frac{10}{3} \\
&< \delta_\epsilon \frac{10}{3} \\
&< \epsilon\frac{3}{10}\frac{10}{3} = \epsilon \\
& && (4)
\end{aligned}$$

Step (3) follows from the fact that the smallest values for x or p in the domain is 1 and this makes the expression as large as possible. $\square$