Theorem Hints
Math 5200/6200.

Theorem 7.3. Suppose that f is continuous on the interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.

Hint: Let $\epsilon > 0$. Since f is continuous on $[a, b]$ for each $p \in [a, b]$ there is a number δ_p so that if $|x - p| < \delta_p$ then $|f(x) - f(p)| < \frac{\epsilon}{2}$. [I think this works, but it could be $\frac{\epsilon}{4}$ or smaller that will be needed.]

Let

$$G = \left\{ \left(p - \frac{\delta_p}{2}, p - \frac{\delta_p}{2} \right) |p \in [0, 1] \right\}.$$

Then since $[a, b]$ is compact, some finite subcollection G' of G covers $[a, b]$. Let

$$G' = \left\{ \left(p_i - \frac{\delta_{p_i}}{2}, p - \frac{\delta_{p_i}}{2} \right) \right\}_{i=1}^{N}.$$

If you pick $\delta = \min\{\frac{\delta_{p_i}}{2} \mid i = 1, \ldots N\}$ then this δ should work for ϵ.

Theorem 6.5. Suppose that f is a continuous function and M is a compact subset of the domain of f. Then $f(M)$ is compact.

Hint. Suppose that G is a collection of segments covering the set $f(M)$. Then for each $x \in f(M)$ there is an element g_x containing x and a segment $S_x = (x - \epsilon_x, x + \epsilon_x)$ containing x and lying in g_x. Some point m_x of M is mapped onto x by f: $f(m_x) = x$. By continuity there is a number δ_x so that f maps $(m_x - \delta_x, m_x + \delta_x)$ into S_x [make sure you know why.] Consider the following collection:

$$J = \{(m_x - \delta_x, m_x + \delta_x) \; |x \in f(M)\}.$$

This is a covering of M, so a finite subcollection J' covers M. Use J' to get a finite subcollection of G that covers $f(M)$.

[Note: this theorem also follows fairly easily from thms 5.2 and 6.3 - which I don't think we've proven yet.]