Anotes04
Applications of the Completeness Axiom.

The completeness of the real numbers; the real numbers $\mathbb{R}$ is a set that satisfies the axioms of arithmetic together with the completeness axiom stated below.

Axiom C (completeness axiom). If $M \subset \mathbb{R}$ and there is an upper bound for M, then there is a least upper bound for M.

I will also refer to Axiom C as the Least Upper Bound axiom.

Exercise 4.1.

a. Show that the set $M = \{\frac{n}{m} | n, m \in \mathbb{Z}^+, \frac{n^2}{m^2} < 2\}$ has a least upper bound L.

b. Show that $L^2 = 2$.

Hint for Ex. 4.1.

Definition. The set H is said to be dense in the set M if and only if every segment containing a point of M also contains a point of H.

Lemma. The set $\{\frac{n^2}{m^2} | n, m \in \mathbb{Z}^+\}$ is dense in $\mathbb{R}^+$.

Exercise 4.2. An alternate completeness axiom is the "Dedekind Cut" axiom: Suppose that the real numbers $\mathbb{R}$ is partitioned into two sets L and R (thus $\mathbb{R} = L \cup R$) so that every point of L is less than every point of R. Then there is either a biggest number in L or a smallest number in R.

Show that this axiom is equivalent to the Least Upper Bound completeness axiom.

Theorem 4.1. If $M \subset \mathbb{R}$ and there is a lower bound for M, then there is a greatest lower bound for M.

Definitions. The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be an increasing sequence if and only if for each positive integer n, $x_n < x_{n+1}$.

The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be a non-decreasing sequence if and only if for each positive integer n, $x_n \leq x_{n+1}$.

Definition. The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be bounded above if and only if there exists a number B so that for each positive integer n, $x_n \leq B$.

The terms "decreasing," "non-increasing" and "bounded below" are similarly defined.

Theorem 4.2. Suppose that $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above. Then the sequence has a sequential limit.

Definition. A set is said to be bounded if it is bounded above and bounded bellow.

Theorem 4.3. If M is an infinite bounded set, then M has a limit point.

Definition. The collection of sets G is said to be monotonic if and only if for each pair of elements g_1 and g_2 either $g_1 \subset g_2$ or $g_2 \subset g_1$.

Exercise 4.3. Find an infinite collection of sets that is monotonic and one that is not monotonic. Find an infinite collection of sets so that no infinite subcollection is monotonic.

Theorem 4.4. Suppose that G is a monotonic collection of intervals. Then there is a point common to all the elements of G.

Exercise 4.4. Find an example of a monotonic collection of sets so that there is no point common to all the elements of the collection.