Anotes06, Continuous Functions.

Definitions and Notation.

The Cartesian plane $\mathbb{R}^2$ is the set of all pairs: $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$; these pairs are called points of the plane.

The set h is a vertical line means that there is a number r so that $h = \{(x, y)|x = r, y \in \mathbb{R}\}$. This line is sometimes denoted by $x = r$.
The set α is a horizontal line means that there is a number a so that $\alpha = \{(x, y)|y = a, x \in \mathbb{R}\}$. This line is sometimes denoted by $y = a$.

A function f is a subset of R^2 such that each vertical line intersects f in at most one point. If the vertical line $x = x_0$ intersects f then $f(x_0)$ denotes the number so that $(x_0, f(x_0))$ is that point of intersection.

The domain of f is the set of all numbers $\{x|(x, y) \in f\}$ and the range of f is the set of all numbers $\{y|(x, y) \in f\}$.

Definition. The function f is *continuous* at the point $(p, f(p))$ means that if $\epsilon > 0$, then there exists a number $\delta > 0$ so that if x is in the domain of f and $|p - x| < \delta$ then $|f(p) - f(x)| < \epsilon$. [Note that typically δ will depend on ϵ and the point p.]
Definition. The function f is said to be *continuous* if it is continuous at each of its points.

Exercise 6.1. A geometric equivalence to continuity: Show that the function f is continuous at the point $P = (x, y)$ if and only if $(x, y) \in f$ and for each pair of horizontal lines α and β with P between there exists a pair of vertical lines h and k with P between them so that that every point of f between h and k also lies between α and β.

Exercise 6.2. In each case also determine the domain and range of the described function.
a.) Show that the function defined by $f(x) = x$ is continuous.
b.) Show that the function defined by $f(x) = x^2$ is continuous.
c.) Show that the function defined by $f(x) = \frac{1}{x}$ when $x \neq 0$ is continuous.

d.) Let $c \in R$, show that the function defined by $f(x) = c$ is continuous. (This is called the constant function.)

e.) Show that the function defined below is continuous:

$$f(x) = \begin{cases} 0 & \text{if } x = -1 \\ \frac{1}{2} & \text{if } x = 1 \\ & \text{is undefined elsewhere} \end{cases}$$

f.) Show that the function defined below is not continuous:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

g.) Show that the function defined below is not continuous:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

h.) Show that the function defined below is continuous at the point $(0, 0)$:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

i.) Show that the function defined below is not continuous at each of its points (this is sometimes called the "salt and pepper" function):

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Definition. If each of f and g is a function and the domain of f is equal to the domain of g then:

$f + g$ denotes the function so that $(f + g)(x) = f(x) + g(x)$;
fg denotes the function so that $(fg)(x) = f(x)g(x)$;
$\frac{f}{g}$ denotes the function so that $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for all x so that $g(x) \neq 0$.

Theorem 6.1 Suppose that each of f and g is a continuous function and the domain of f is equal to the domain of g then:

a. $f + g$ is continuous;
b. fg is continuous;
c. $\frac{f}{g}$ is continuous at each point where $g(x) \neq 0$.

Unless otherwise stated (explicitly or implicitly) assume that all the functions in the following theorems have domain all the reals.

Theorem 6.2. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ has sequential limit p and that f is a function that is continuous at the point $(p, f(p))$. Then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ has sequential limit $f(p)$.

Definition. If $f : \mathbb{R} \to \mathbb{R}$ is a function and $M \subset \mathbb{R}$ then $f(M)$ denotes the set $\{f(x)|x \in \mathbb{R}\}$ and $f^{-1}(M) = \{x|f(x) \in M\}$.

Theorem 6.3. The function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for each open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is open.

Definition. Suppose that f and g are functions so that the domain of f is equal to the range of g. Then $f \circ g$ is the function defined by:

$$(f \circ g)(x) = f(g(x)).$$

Theorem 6.4. Suppose that f and g are continuous functions so that the domain of f is equal to the range of g. Then $f \circ g$ is continuous.

Theorem 6.5. Suppose that f is a continuous function and M is a compact subset of the domain of f. Then $f(M)$ is compact.

[Hint: use theorem 6.3.]

Exercise 6.3. Determine which of the following are true,

a. If f is a function, M is a subset of the domain of f and p is a limit point of M then $f(p)$ is a limit point of $f(M)$.

b. If f is a continuous function, M is a subset of the domain of f and p is a limit point of M then $f(p)$ is a limit point of $f(M)$.

c. If f is a function, M is a subset of the range of f and p is a limit point of M then $f^{-1}(p)$ is a limit point of $f^{-1}(M)$.

d. If f is a continuous function, M is a subset of the range of f and p is a limit point of M then $f^{-1}(p)$ is a limit point of $f^{-1}(M)$.

Cauchy Sequences.

Definition. Suppose that $X = \{x_n\}_{n=1}^{\infty}$ is a sequence. Then X is said to be a Cauchy sequence if and only if for each $\epsilon > 0$ there exists an integer N so that if $n, m > N$ then $|x_n - x_m| < \epsilon$.

Exercise 6.4. Show that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 6.5. Show that if $X = \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and the set $M = \{x_n | n \in \mathbb{Z}^+\}$ is finite then there is a term x_k of the sequence so that all the terms after the x_k^{th} term is equal to x_k.

Definition. The sequence $X = \{x_n\}_{n=1}^{\infty}$ is said to converge if it has a sequential limit point and to diverge if it does not.

If the sequence $\{x_n\}_{n=1}^{\infty}$ converges then the sequential limit is denoted by

$$\lim_{n \to \infty} x_n.$$

Theorem 6.6. If the sequence $X = \{x_n\}_{n=1}^{\infty}$ converges, then it is a Cauchy sequence.

Theorem 6.7. If $X = \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then it converges.

Exercise 6.6. Consider the following "Axiom" of the reals.

Axiom CC: *Every Cauchy sequence converges.*

Show that Axiom CC is equivalent to the least upper bound axiom.